

Contents

Preface — V

Introduction — VII

1	Basic notation and preliminaries — 1
1.1	Complex Hilbert and Banach spaces — 1
1.2	Linear operators and their adjoints — 7
1.2.1	Adjoint and self-adjoint operators — 8
1.2.2	Bounded linear operators — 9
1.3	Positive operators — 13
1.4	Resolvent set and spectrum — 15
1.5	Unitary operators — 16
1.6	Projection operators — 17
1.7	Finite rank and compact linear operators — 21
1.7.1	Finite rank linear operators — 21
1.7.2	Compact linear operators — 22
1.8	Trace-class and Hilbert–Schmidt operators — 25
1.8.1	Trace-class operators — 25
1.8.2	Hilbert–Schmidt operator — 29
2	Formulation of quantum systems — 33
2.1	Operator topologies — 34
2.2	Operator algebras — 36
2.2.1	C^* -algebras — 37
2.2.2	von Neumann algebras — 39
2.3	Bounded linear functionals — 42
2.3.1	Bounded linear functionals on $\mathbb{H}$ — 44
2.3.2	$\mathcal{A}^*$ and $\mathcal{A}_*$ — 48
2.4	Quantum states — 53
2.5	GNS representation — 56
2.6	Quantum observables and measurements — 58
2.6.1	Positive operator valued measures — 58
2.6.2	Quantum observables — 59
2.6.3	Quantum measurements — 60
2.7	Tensor products and direct sums — 62
2.7.1	Tensor products of Hilbert spaces and operators — 62
2.7.2	Direct sum of Hilbert spaces and operators — 65
2.8	Composite quantum systems — 67
2.8.1	Composite system as tensor products — 67
2.8.2	Partial traces — 67

3	Probability measures and convex functions on $\mathcal{S}(\mathbb{H})$ — 71
3.1	Probability measures on $\mathcal{S}(\mathbb{H})$ — 71
3.1.1	Support of Borel measures — 71
3.2	Some compactness criteria — 74
3.2.1	Compactness criteria on $\mathcal{S}(\mathbb{H})$ — 74
3.2.2	Compactness criteria on $\mathcal{P}(\mathcal{S}(\mathbb{H}))$ — 79
3.3	Barycenters — 80
3.3.1	A brief review of convex analysis — 80
3.3.2	Properties of barycenter — 81
3.4	Convex functions on $\mathcal{S}(\mathbb{H})$ — 88
3.4.1	σ -convexity and μ -convexity — 88
3.4.2	Convex hull and convex closure — 91
4	Completely positive maps — 95
4.1	Definitions and properties — 95
4.2	Some technical results — 102
4.3	Stinespring representation — 108
4.4	Kraus representation — 113
5	Quantum channels and operations — 117
5.1	Quantum operations — 117
5.2	Quantum channels — 118
5.2.1	Quantum channels in the Schrodinger picture — 119
5.2.2	Quantum channels in the Heisenberg picture — 120
5.3	Representations and dilations of channels — 123
5.3.1	Stinespring representation — 123
5.3.2	Unitary dilation — 124
5.3.3	Kraus representation — 127
5.4	Structural properties of quantum channels — 128
5.4.1	Convexity of channels — 128
5.4.2	Concatenation of channels — 128
5.4.3	Tensor product of channels — 129
5.5	Purification — 129
5.6	Reversible quantum channels — 130
5.7	Complementary channels — 132
6	Approximation and convergence of quantum channels — 133
6.1	Distinguishability of quantum states — 133
6.1.1	Fidelity of quantum states — 133
6.1.2	Bures distance — 145
6.2	Channel fidelity — 145
6.2.1	Monotonicity of channel fidelity — 146

6.2.2	Ensemble average and entanglement fidelities — 146
6.3	Norms on unconstrained channels — 149
6.3.1	Diamond norm — 149
6.3.2	Complete boundedness norm — 149
6.3.3	Bures distance — 151
6.4	Norms on constrained channels — 153
6.4.1	Energy-constrained operator norm — 154
6.4.2	Constrained diamond norms and Bures distances — 156
6.5	Approximations of quantum channels — 159
6.6	Convergences of channels — 161
6.6.1	Strong and uniform convergence — 161
6.6.2	Strong* convergence of channels — 165
7	von Neumann entropy — 169
7.1	von Neumann entropy on $\mathcal{S}(\mathbb{H})$ — 169
7.2	Properties of $H(\cdot)$ on $\mathcal{S}(\mathbb{H})$ — 171
7.3	$H(\cdot)$ on subsets of $\mathcal{S}(\mathbb{H})$ — 177
7.3.1	Closed convex subsets $\mathcal{A} \subset \mathcal{S}(\mathbb{H})$ — 177
7.3.2	$\mathcal{K}_{\mathbb{H}}(h)$ — 178
7.3.3	$H(\cdot)$ on an arbitrary closed subset $\mathcal{K}$ — 192
7.4	$H(\cdot)$ extended to $\mathfrak{T}_+(\mathbb{H})$ — 196
7.4.1	Properties of $H(\cdot)$ on $\mathfrak{T}_+(\mathbb{H})$ — 197
7.4.2	Discontinuity of extended quantum entropies — 200
8	Relative and conditional entropies — 203
8.1	Quantum relative entropy — 203
8.2	$H(\cdot \ \cdot)$ extended to $\mathfrak{T}_+(\mathbb{H}) \times \mathfrak{T}_+(\mathbb{H})$ — 210
8.3	Quantum conditional entropy — 213
8.3.1	Definitions — 213
8.3.2	Properties of conditional entropy — 214
9	Channel output entropies — 223
9.1	Monotonicity of output relative entropy and Petz's theorem — 223
9.1.1	Monotonicity of output relative entropy — 223
9.1.2	Petz's theorem on output relative entropy — 225
9.2	Continuity of output entropies — 228
9.2.1	Positive linear map Φ — 228
9.2.2	Quantum operations — 234
9.3	Convex closure of output entropy — 239
10	Quantum entanglement — 243
10.1	Schmidt decomposability of states — 243

10.2	Separability of states — 250
10.2.1	Matrix of amplitudes — 250
10.2.2	Separability — 255
10.3	Measurements of entanglement — 263
10.3.1	Entropy of entanglement — 266
10.3.2	Relative entropy of entanglement — 269
10.3.3	Entanglement of formation — 272
11	Quantum mutual and coherent information — 277
11.1	Quantum mutual information — 277
11.1.1	Finite-dimensional case — 284
11.1.2	Infinite-dimensional case — 286
11.2	Coherent information — 288
12	Holevo χ-capacity — 295
12.1	The χ -functions — 295
12.1.1	Input χ -function — 295
12.1.2	Output χ -function — 296
12.2	Holevo χ -capacities — 303
12.2.1	Optimal ensembles — 305
12.2.2	Compact constraint $\mathcal{A}$ — 312
12.2.3	Convex constraint $\mathcal{A}$ — 313
12.3	Continuity of Holevo χ -capacity — 320
12.4	Holevo χ -quantity and coherent information — 324
12.5	Additivity of Holevo χ -capacities — 327
12.5.1	Hastings' counterexamples to additivity — 328
12.5.2	Additivity of unconstrained χ -quantity — 329
12.5.3	Tensor product of constrained channels — 334
12.5.4	Additivity of constrained Holevo χ -capacity — 335
13	Classical capacities of memoryless channels — 339
13.1	Transmissions of classical information — 341
13.1.1	Preparation of classical information — 341
13.1.2	Encoding and decoding of classical information — 342
13.2	Unconstrained classical capacity — 346
13.2.1	Holevo–Schumacher–Westmoreland theorem — 347
13.2.2	Quantum version of Feinstein's lemma — 348
13.3	Classical capacity of constrained channels — 355
13.4	Entanglement-assisted classical capacity — 358
13.4.1	Finite-dimensional channels — 360
13.4.2	Constrained infinite-dimensional channels — 367
13.4.3	Continuity of $C_{\text{ea}}(\cdot) : \mathfrak{Q}\mathfrak{C}(A, B) \rightarrow [0, +\infty]$ — 374

13.5	Comparison of classical capacities —	377
13.5.1	Finite-dimensional unconstrained capacities —	377
13.5.2	Infinite-dimensional constrained capacities —	379
14	Structure of quantum memory channels —	383
14.1	Representations of memoryless channels —	384
14.1.1	Kraus representation in the Schrodinger picture —	384
14.1.2	Kraus representation in the Heisenberg picture —	385
14.1.3	Unitary representation in the Schrodinger picture —	385
14.2	Constructive approach to memory channels —	386
14.2.1	Unitary representation in the Schrodinger picture —	389
14.2.2	Unitary representation in the Heisenberg picture —	390
14.2.3	Kraus representation of the memory channel —	391
14.3	Quasilocal algebras approach —	391
14.3.1	Construction of quasilocal algebras —	391
14.3.2	Structure of causal channels —	394
14.4	Forgetful channels —	397
14.4.1	Definition of forgetful channels —	397
14.4.2	Topological properties —	404
15	Channels with Markovian memory —	405
15.1	A brief review on Markov chains —	405
15.2	Constructions of Markov memory models —	407
15.3	Channels with ergodic Markovian memory —	409
15.3.1	Classical capacity —	409
15.4	Channels with general Markov memory —	422
15.4.1	Preamble for Markovian memory —	422
15.4.2	Classical capacity —	429
16	Channels with long-term memory —	441
16.1	The model —	441
16.2	Classical product state capacity —	442
16.2.1	Infinite-dimensional classical capacity —	443
16.2.2	Constrained classical capacity in infinite dimensions —	455
16.3	Entanglement-assisted classical capacity —	458
16.3.1	Unconstrained case —	461
16.3.2	Energy constrained case —	466

Bibliography — 471

Index — 479