

Shahid Rahman · Zoe McConaughey  
Ansten Klev · Nicolas Clerbout

# Immanent Reasoning or Equality in Action

A Plaidoyer for the Play Level

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Shahid Rahman • Zoe McConaughey  
Ansten Klev • Nicolas Clerbout

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Springer

Shahid Rahman  
Univ. Lille  
Département de philosophie  
CNRS, UMR 8163 - STL - Savoirs  
Textes Langage  
Lille, France

Ansten Klev  
Institute of Philosophy  
Czech Academy of Sciences  
Prague, Czechia

Zoe McConaughey  
Univ. Lille  
CNRS, UMR 8163 - STL - Savoirs  
Textes Langage  
Lille, France

Département de philosophie &  
Centre Interuniversitaire de Recherche en  
Science et Technologie  
Université du Québec à Montréal  
Montréal, QC, Canada

Nicolas Clerbout  
Universidad de Valparaíso  
Valparaíso, Chile

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*To all of the present and past members of the  
Dialogicians' team of Lille and beyond*

# Preface

Prof. Göran Sundholm of Leiden University inspired the group of logicians who nowadays develop their work in Lille and Valparaíso to undertake a fundamental review of the dialogical conception of logic by linking it to Constructive Type Logic. One of Sundholm's insights was that inference can be understood as involving an *interlocutor*. This led to several investigations whose purpose was to explore the consequences of joining winning strategies to the proof-theoretical conception of meaning: while introduction rules lay down the conditions under which a winning strategy for the Proponent can be built, the elimination rules lay down precisely those elements of the Opponent's assertions that the Proponent has the right to use for building a winning strategy. The pragmatic and ethical features of obligations and rights naturally bring forth the dialogical interpretation of natural deduction.

During the 2012 Visiting Professorship of Prof. Sundholm in Lille, the logic group of Lille started probing possible ways of implementing Per Martin-Löf's Constructive Type Theory (CTT)<sup>1</sup> in the dialogical perspective. The first publication in particular on the subject—Aarne Ranta's (1988) paper—was read and discussed during Sundholm's seminar. These discussions strongly suggested that the game-theoretical conception of quantifiers, which marshalls interdependent moves, provides a natural link between CTT and dialogical logic. This idea triggered several publications by the group of Lille in collaboration with Nicolas Clerbout and Juan Redmond at the University of Valparaíso, including that of the (2015) book by Clerbout and Rahman providing a systematic development of this way of linking CTT and the dialogical conception of logic.

However, the (Clerbout & Rahman, 2015) book was written from the CTT perspective on dialogical logic, rather than the other way round. The present book, *Immanent Reasoning or Equality in Action*, should provide the other perspective in the dialogue between the dialogical framework and Constructive Type Theory.

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<sup>1</sup>For an overview, see for instance (Nordström, Petersson, & Smith, 1990, 2000), (Primiero, 2008), (Thompson, 1991).

In order to develop a dialogical perspective on the links between CTT and dialogical logic, we will follow three complementary paths:

- A. One of the chief ideas animating our study is that we believe Sundholm's (1997)<sup>2</sup> notion of *epistemic assumption* is closely linked to the *Copy-cat rule* or *Socratic rule* that distinguishes the dialogical framework from any other game-theoretical approach; this link is established through the dialogical understanding of definitional equality.
- B. We will join—with some nuances linked to point C below—Martin-Löf's (2017a, 2017b) suggestions that the new insights provided by the dialogical framework mainly amount to the following three interconnected points:
  - B.1. The introduction of *rules of interaction* rather than of inference rules.
  - B.2. The challenge to what Kuno Lorenz (2010a, p. 71) calls the *semantization of pragmatics*: deontic features are formalized with the help of specific propositional operators (and indexes) upon which the truth-value of the resulting proposition is made dependent.
  - B.3. The central role of the notion of *execution* in the rules of interaction: executions are responses to questions of *knowing how*.
- C. As indicated by the subtitle, “A Plaidoyer for the Play Level,” we will stress the importance of the play level over the strategy level: this binds the point of execution with that of equality.

In relation to A and B.3, the present book can indeed be read as furthering Sundholm's own extension to inference of Austin's remark (1946, p. 171) on assertion acts; Sundholm (2013a, p. 17) did indeed produce this forceful formulation:

*When I say therefore, I give others my authority for asserting the conclusion, given theirs for asserting the premisses.*

In recent lectures, Per Martin-Löf used the dialogical perspective with epistemic assumptions in order to escape a form of circle threatening the explanation of the notions of inference and demonstration. A demonstration may indeed be explained as a chain of (immediate) inferences starting from no premisses at all. That an inference

$$\frac{J_1 \dots J_n}{J}$$

is valid means that one can make the conclusion (judgement  $J$ ) evident on the assumption that  $J_1, \dots, J_n$  are known. Thus, the notion of epistemic assumption appears when explaining what a valid inference is. According to this explanation, however, we cannot take “known” in the sense of *demonstrated*, or else we would be explaining the notion of inference in terms of demonstration when demonstration has

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<sup>2</sup>See also (Sundholm, 1998, 2012, 2013b).



been explained in terms of inference. Hence, the threatening circle. In this regard, Martin-Löf suggests taking “known” here in the sense of *asserted*, which yields epistemic assumptions as judgements others have made, judgements whose responsibility others have already assumed. An inference being valid would accordingly mean that, given others have assumed responsibility for the premisses, I can assume responsibility for the conclusion:

Martin-Löf’s circularity problem

*The circularity problem is this: if you define a demonstration to be a chain of immediate inferences, then you are defining demonstration in terms of inference. Now we are considering an immediate inference and we are trying to give a proper explanation of that; but, if that begins by saying: Assume that  $J_1, \dots, J_n$  have been demonstrated—then you are clearly in trouble, because you are about to explain demonstration in terms of the notion of immediate inference, hence when you are giving an account of the notion of immediate inference, the notion of demonstration is not yet at your disposal. So, to say: Assume that  $J_1, \dots, J_n$  have already been demonstrated, makes you accusable of trying to explain things in a circle. The solution to this circularity problem, it seems to me now, comes naturally out of this dialogical analysis. [...]*

*The solution is that the premisses here should not be assumed to be known in the qualified sense, that is, to be demonstrated, but we should simply assume that they have been asserted, which is to say that others have taken responsibility for them, and then the question for me is whether I can take responsibility for the conclusion. So, the assumption is merely that they have been asserted, not that they have been demonstrated. That seems to me to be the appropriate definition of epistemic assumption in Sundholm’s sense.<sup>3</sup>*

The present study makes a further step, namely that of relating judgemental equality with the rule known in the literature as the *Copy-cat rule*, or *Formal rule*, or, as more aptly called now by Marion & Rückert (2015) the *Socratic rule*.<sup>4</sup> We hold it as one of our main tenets that this relation provides both a simpler and a more direct way to implement the Constructive Type Theoretical approach within the dialogical framework. Such a reconsideration of the Socratic rule roughly amounts to the following:

1. When Proponent **P** makes a move bringing forward a local reason—say  $b$ —to defend an elementary proposition  $A$ , this move can be challenged by the Opponent **O**. That is, given  $\mathbf{P} b : A$ , the antagonist may play the attack  $\mathbf{O} ? = b$ .
2. To respond to such a challenge from **O**, **P** must bring forward a definitional equality expliciting that the local reason chosen by **P** copies precisely the reason **O** chose when stating  $A$ . In short, this equality expresses at the object language level the fact that **P**’s defence move rests on the authority **O** has previously asserted when producing her local reason.

More generally, according to this view a definitional equality established by **P** and brought forward while defending the proposition  $A$  expresses the equality between a local reason (introduced by **O**) on the one hand and the instruction on the other hand used for building a local reason brought forward by **O** when stating  $A$ . A definitional

<sup>3</sup>Transcription by Ansten Klev of Martin-Löf’s talk in May 2015 (Martin-Löf, 2015).

<sup>4</sup>See for instance below, Sect. 3.2.2 or Sect. 7.2.1.

equality can therefore be read as a computation rule indicating how to compute the instructions **O** brought forward during a play.<sup>5</sup>

From the dialogical perspective though, providing local reasons must be distinguished from providing equalities: while providing an explicit local reason  $b$  is a way of answering a *why* question, such as “why does  $A$  hold?”, providing an equality is more a way of answering a *how* question, such as “how do you show that  $b$  accomplishes the explicative task?”. Equalities thus express how to execute or carry out the actions encoded by the local reasons.

Let us recall that from the strategic point of view, **O**’s moves correspond to elimination rules (including the selector-functions deployed by these rules) of demonstrations. Thus, the dialogical rules prescribing how to introduce a definitional equality correspond—at the strategy level—to the definitional equality rules for CTT as applied to the selector-functions involved in the elimination rules.

We are in this fashion extending the dialogical interpretation of Sundholm’s *epistemic assumption* to the rules that set up the definitional equality of a type. Actually, Sundholm (2017) himself suggests in his section 4 this extension when he points out that if some object, say  $a$ , is granted by a suitable *epistemic assumption* to be a proof-object of  $C$ , then it *executes to a canonical proof of C*. In other words, on the grounds of the epistemic assumption we know that  $a$  must be equal to a canonical element of  $C$ .

Notice however that from the dialogical perspective equalities grounded on the sole authority of the Opponent (i.e., on epistemic assumptions) are a trademark of what we call *formal dialogues*.

Yet, the dialogical perspective also includes *material dialogues*, where the Opponent must carry out some process *specific* to the proposition at stake before the Proponent can answer to the *how*-question with a suitable equality. In other words, though equalities of material dialogues are the result of the application of the Socratic rule, they are not “merely” grounded on epistemic assumptions.

In relation to **B1** and **B2**, the Oslo and Stockholm lectures of Martin-Löf (2017a, 2017b) contain challenging and deep insights in dialogical logic, and the understanding of *defences as duties* and *challenges as rights* is indeed at the core of the deontic feature underlying the dialogical framework. More precisely, these two rules *Req1* and *Req2*:

$$(Req\ 1) \frac{\vdash C}{?\vdash_{may} C}$$

and

$$(Req\ 2) \frac{\vdash C \quad ?\vdash C}{\vdash_{must} C'}$$

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<sup>5</sup>These elements are formalized in the Socratic rule for immanent reasoning, Sect. 7.2.1.

both condense the particle rules of meaning and bring to the fore the normative feature of those rules. What is more, Martin-Löf points out rightly that they should not be called *rules of inference* but *rules of interaction*.

Still, a dialogician might wish to draw further distinctions to the divide between play level rules and those of the strategy level, such as distinctions between players or the distinction in terms of choice as to how to defend or challenge moves: it is such a distribution of choices that distinguishes the meaning for instance of the conjunction and of the disjunction; the meaning of a disjunction binds the right to state a disjunction with the *defender's duty to choose* a component of the disjunction to defend, but the meaning of the conjunction binds the right to challenge it with the *challenger's duty to choose* the side to be requested.<sup>6</sup>

On our view, point **C** is at the core of the innovations of the dialogical framework and our point of departure from Ranta's (1988) seminal paper: he proposes to identify proof-objects with winning strategies, so that we have canonical and non-canonical winning strategies. Winning strategies are however not primitive in the dialogical framework, but are constituted by some finite sequence of legal moves (that is, a sequence of moves which observes the game rules) called plays. The notion of plays is what grounds meaning within the dialogical framework, and this notion also leads to the notion of proposition: in the standard presentation of dialogical logic a proposition is defined as a *dialogue-definite expression*, that is, an expression  $A$  such that there is an individual play about  $A$  that can be said to be lost or won after a finite number of steps, following some given rules of dialogical interaction.

As discussed in Chap. 3 and Sect. 11.1, the rock-bottom of the dialogical approach to CTT is the play level notion of dialogue-definiteness of the proposition. Thus to paraphrase Lorenz (2001, p. 258): *for an expression to count as a proposition  $A$  there must exist an individual play about the statement  $\mathbf{X} ! A$ , in the course of which  $\mathbf{X}$  is committed to bring forward a local reason to back that proposition, play which must reach a final position with either win or loss after a finite number of moves according to definite particle and structural rules.*

Though performing the interaction schemata defining a play is in this sense a crucial aspect of the dialogical framework, it must be stressed that the actualization of a play (performing it) *does not* require winning the play. Immanent reasoning thus conceives *performance* as *putting dialogue-definiteness into action*.

In a nutshell, we call our dialogues involving rational argumentation *dialogues for immanent reasoning* precisely because the *reasons* backing a statement, now *explicit denizens* of the object language of plays, *are internal* to the development of the dialogical interaction itself: the emergence of concepts are not only games of

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<sup>6</sup>In the conclusion (Sect. 11.5), we enrich Martin-Löf's (2017a, 2017b) rules Req1 and Req2 with players and with choice-options.

giving and asking for reasons (games involving *why*-questions), they are also games including moves establishing *how is it that the reason brought forward accomplishes the explicative task*. Immanent reasoning is thus a dialogical framework for games of *why* and *how*.

Lille, France

Lille, France and Montréal, Québec

Prague, Czech Republic

Valparaíso, Chile

Shahid Rahman

Zoe McConaughy

Ansten Klev

Nicolas Clerbout

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Univ. Lille, CNRS, UMR 8163 – STL –  
Savoirs Textes Langage, Lille, France

Shahid Rahman

Univ. Lille, CNRS, UMR 8163 – STL –  
Savoirs Textes Langage, Lille, France

Zoe McConaughy

Université du Québec à Montréal,  
Montreal, QC, Canada

Academy of Sciences of the Czech  
Republic, Prague, Czech Republic

Ansten Klev

Instituto de Filosofía  
Universidad de Valparaíso,  
Valparaíso, Chile

Nicolas Clerbout

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# Chapter 1

## Introduction: Some Brief Historical and Philosophical Remarks



The present volume develops a new way of linking Constructive Type Theory (CTT) with dialogical logic by following these three complementary paths, as mentioned in the preface:

- A. The path observing that Sundholm's (1997)<sup>1</sup> notion of *epistemic assumption* is closely linked to the *Copy-cat* and *Socratic rules*<sup>2</sup> and that it provides the dialogical conception of definitional equality;
- B. the path joining (in principle) Martin-Löf in his (2017a, 2017b) suggestions, according to which the new insights provided by the dialogical framework mainly amount to the following three interconnected points:
  - B.1. the introduction of *rules of interaction* rather than of rules of inference;
  - B.2. the challenge to the *semantization of pragmatics* and the claim of the deontic nature of logic;<sup>3</sup>
  - B.3. the central role of the notion of *execution* in the rules of interaction: executions are responses to questions of *knowing how*.
- C. The path stressing the importance of the *play level* and the associated notion of *dialogue-definiteness*.

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<sup>1</sup>See also (Sundholm, 1998, 2012, 2013).

<sup>2</sup>See Sect. 3.2.2 for the Copy-cat rule, and Sect. 7.2.1 for the Socratic rule.

<sup>3</sup>In fact, as opposed to Martin-Löf's understanding of dialogical logic, Lorenz's dialogical constructivism does not only reject the *semantization of pragmatics* in which deontic features are formalized using specific propositional operators and indexes upon which depends the truth-value of the resulting proposition, but it also rejects the *pragmatization of semantics* in which a propositional kernel is complemented by moods yielding assertions, questions, commands, and so on. According to dialogical constructivism, pragmatic and semantic features are produced within one and the same act. See (Lorenzen, 1969), (Kamlah & Lorenzen, 1972), (Lorenzen & Schwemmer, 1975). It is precisely this tenet on the dual nature of actions in both their *significative* and *communicative* role, thoroughly worked out by Lorenz (2010a, pp. 71–80), that leads to this central claim that logic is part of ethics—see Sect. 11.5 for further details.

Before displaying some of the conceptual background behind the project we will first make some brief historical remarks concerning the *dialogical turn* that took Lorenzen (1955) from his *Operative Logik* to the inception of *dialogical logic*.

## 1.1 The Dialogical Turn and the Operative Justification of Intuitionistic Logic

The origins of a deontic nature of logic in its dialogical conception can be traced back to Paul Lorenzen's 1958 endeavour to overcome difficulties specific to his *Einführung in die Operative Logik und Mathematik* (1955), which lead him to turn the normative perspectives of the operative logic into the dialogical framework. We will here closely follow Schröder-Heister's thorough (2008) paper on the subject.<sup>4</sup> It should be noted that these difficulties are reminiscent of Martin-Löf's circularity puzzle mentioned in the Preface and which motivated his dialogical interpretation of the notion of epistemic assumption (see Preface).

### 1.1.1 Admissibility in Operative Logic

In the context of the operative justification of intuitionistic logic, the operative meaning of an elementary proposition is understood as a proof of its derivability in relation to some given calculus. Calculus is here understood as a general term close to the *formal systems* of Curry (1951) which include basic expressions, and rules for producing complex expressions out of basic ones. More precisely, as Schröder-Heister puts it:

*Lorenzen starts with elementary calculi (OL, §1) which permit to generate words (strings of signs) over an arbitrary (finite) alphabet. The elements of the alphabet are called atoms, the words are called sentences ("Aussagen"). A calculus  $K$  is specified by giving certain initial formulas ("Anfänge")  $A$  and rules  $A_1, \dots, A_n \rightarrow A$ .<sup>5</sup>*

Instead of starting with the functor-argument structure common in logic, Lorenzen starts here with an arbitrary word-structure, where expressions in  $K$  are just strings of atoms and variables, allowing his notion of calculus to be particularly general.

In such a framework, logic is introduced as a system of proof procedures for asserting the *admissibility* of rules.<sup>6</sup>

<sup>4</sup>See also Lorenz's (2001) study of the origins of the dialogical approach to logic.

<sup>5</sup>(Schröder-Heister 2008). All the following quotations of this section, if not otherwise specified, will come from this same source.

<sup>6</sup>Nowadays, the notion of *admissibility* is a fundamental concept of proof-theory; Schröder-Heister (2008, p. 218) pointed out that Lorenzen was the one to have coined this term.

A rule  $R$  is called *admissible* in a calculus  $K$ , if its addition to the primitive rules of  $K$ —resulting in an extended calculus  $K + R$ —does not enlarge the set of derivable sentences. If  $\vdash_K A$  denotes the derivability of  $A$  in  $K$ , then  $R$  is admissible in  $K$  if

$$\vdash_{K+R} A \text{ implies } \vdash_K A$$

for every sentence  $A$ .

Thus, implication is explained in terms of admissibility. But how is admissibility to be explained?

### 1.1.2 Implication and Admissibility: Another Circle?

Since implication is explained by the notion of admissibility, admissibility cannot be explained by the notion of implication. In fact, Lorenzen (1955), in his chapter 3, invests admissibility with an operative meaning through the notion of an *elimination procedure*, stating that  $R$  is admissible in  $K$  if every application of  $R$  can be eliminated from every derivation in  $K + R$ . The above implication ( $\vdash_{K+R} A$  implies  $\vdash_K A$ ) reduces to a form of elimination procedure, for the derivation in  $K + R$  can be brought down to a derivation which no longer uses  $R$ : the  $R$  rule can be disposed of. Schröder-Heister thus concludes:

*According to Lorenzen, this is the sort of insight (evidence) on which constructive logic and mathematics is based. It goes beyond the insight that something is derivable in  $K$ , but is still something which has a “definite” meaning.<sup>7</sup>*

This approach thus goes beyond the formalistic focus on derivability: what provides meaning is *the further understanding gained through the notion of admissibility*. In this respect, according to Schröder-Heister:

*Lorenzen’s theory of implication is based on the idea that an implicational sentence  $A \rightarrow B$  expresses the admissibility of the rule  $A \rightarrow B$ , so the assertion of an implication is justified if this implication, when read as a rule, is admissible. In this sense an implication expresses a meta-statement about a calculus. This has a clear meaning as long as there is no iteration of the implication sign.<sup>8</sup>*

It is precisely to deal with iterated implications that Lorenzen develops the idea of finitely iterated meta-calculi. Schröder-Heister (2008, p. 235) points out that the operative approach has its own means to draw the distinction between *direct* and *indirect inferences*, a distinction which triggered Martin-Löf’s puzzle quoted in our preface (see Preface). In this sense, the implication  $A \rightarrow B$  can be asserted as either

- (i) a *direct derivation* in a meta-calculus  $MK$ , based on a demonstration of the admissibility in  $K$  of the rule  $A \rightarrow B$ , or as
- (ii) an *indirect derivation* by means of a formal derivation in  $MK$  using axioms and rules already shown to be valid.

<sup>7</sup>(Schröder-Heister, 2008, p. 217).

<sup>8</sup>(Schröder-Heister, 2008, p. 222).

In the context of operative logic, *direct knowledge* or *canonical inference* of the implication  $A \rightarrow B$  is obtained by the demonstration of the admissibility in  $K$  of the rule  $A \rightarrow B$ , and *indirect knowledge* or *non-canonical inference* results from the derivation of  $A \rightarrow B$  by means of rules already established as admissible.

### 1.1.3 From Admissibility to Dialogue-Definiteness

There is however a high price to pay for this way out of the circularity problem, as knowledge cannot be characterized in the required way showing that the reasoner actually masters the meaning of an implication. Schröder-Heister (2008, p. 236) indeed pointed out that in the Gentzen-style introduction rule for implication, the conclusion prescribes that there is a derivation of the consequent from the antecedent, *independently of the validity of the hypothetical derivation* itself. Indeed, the meaning explanation of the implication is based on the idea that from the assumption of a derivation of the antecedent a method can be found that transforms the derivation in one of the consequent.

This undesired consequence on knowledge motivated Lorenzen to move to the dialogical framework in which the *play level* takes care of all the issues on meaning and *strategies* are associated to validity features: in this context, a proof of admissibility amounts to showing that some specific sequence of plays yields a winning strategy.

Now, if dialogues are to be conceived as mediators of meaning, these dialogues must be games actually playable by human beings: it must be the case that *we can actually perform them*—see our Sect 11.1.<sup>9</sup> These games must therefore be finite, though this does not excluded that there might be a (potentially) infinite number of them. In fact it is the notion of *dialogue-definiteness* that provides both, the basis for implementing the requirement of human-playable games, and the notion of proposition. Under such a background a proposition is defined as a dialogue-definite expression, that is, an expression  $A$  such that there is an individual play about  $A$ , that can be said to be lost or won after a finite number of steps, following some given rules of dialogical interaction.

Notice however that the notion of dialogue-definiteness is not bound to knowing how *to win*—this is rather a feature that characterizes winning strategies—, but to master the meaning of an implication, within the dialogical framework, amounts rather to *know how* to develop *an actual play for it*. In this context it is worth mentioning that during the Stockholm and Oslo talks on dialogical logic, Martin-Löf (2017a, 2017b) points out that one of the hallmarks of the dialogical approach is the notion of *execution*, which—as mentioned in the preface—is close to the requirement of *bringing forward a suitable equality while performing an actual play*. Indeed, from the dialogical point of view (see Sect. 1.2.2), an equality statement

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<sup>9</sup>See also Marion (2006, p. 231).



comes out as an answer to a question on the local reason *b* of the form *how*: *How do you show the efficiency of b as providing a reason for A?* In this sense the *how*-question presupposes that *b* has been brought-forward as an answer to a *why* question: *Why does A hold?* Thus, equalities express the way how to execute or carry out the actions encoded by the local reason; however, the actualization of a play-schema *does not require the ability of knowing how to win a play*. Thus, while execution, or *performance*, is indeed important—see our point **B.3** above—the backbone of the framework lies in the *dialogue-definiteness notion of a play*.

Perhaps a way of formulating the distinction we are aiming at is to stress the difference between *ability* and *knowing how*. In this context, one might speak of *ability* in the sense of the *ability to win*—in a way not far from Peregrin’s (2014, pp. 228–229) notion of *tactics*—, but *ability* has strategy level underpinnings rather than play level ones. The fundamental notion in this dialogical perspective is therefore that of *knowing how to do develop a play* for some proposition *C*, rather than that of having the ability to develop a *winning* play for *C*.

This is how the problematic case of operative logic is overcome by a turn where the actions that were understood as operations within the framework of *operative logic* are now understood as *dialogical interactions*. In other words, *the dialogical approach turned monological operations into dialogical interactions* (see Sect. 11.5).<sup>10</sup>

## Content and Interaction

Another important issue in the passage from the operative to the dialogical framework is that while the operative framework allowed quite naturally to deal with mathematical content, the dialogical framework appears to be restricted to the meaning of logical constants. This has been the subject of many criticisms, old (Hintikka, 1973, pp. 77–82), and new (Trafford, 2017, pp. 86–88); see Chap. 11, in particular Sect. 11.4. There have nonetheless been attempts to compensate this gap by introducing in the dialogical framework definitions conceived as operation rules—see Piecha and Schröder-Heister (2011) and Piecha (2012). However, these attempts have rather been received as highly programmatic.

It is actually quite fair to say that the notion of *material dialogues*—that is dialogues containing rules for expressions other than logical constants—seems to be underdeveloped in respect to *formal dialogues* (restricted mostly to logical constants) which have gathered much more attention. It is also true that a similar kind of criticism has also been raised against inferentialist approaches to meaning,

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<sup>10</sup>Winning strategies in the first writings of Lorenzen and Lorenz (1978) were formulated in the form of sequent-calculus; thus the demonstration of “admissibility” amounts in this context to show that the sequence of plays determined by the local and structural rules for the logical constants yield those of the sequent calculus.

and *operative logic* and dialogical logic, inspired by these inferentialist approaches, seemed to inherit this problem. However, let us stress that the fathers of dialogical logic were aware of the need for a contentual (*material* was the chosen term) basis from the beginning, and they tackled the issue with different devices. Lorenz (1970) in particular dedicated to this issue very thorough and deep studies, most of them collected in (Lorenz, 2010a, 2010b).

One of the widely acknowledged achievements of Constructive Type Theory rests in its ability to furnish the means to develop a language in which mathematical content can be introduced with the same kind of inferential rules displayed by systems of natural deduction. This virtue of CTT motivated us to explore the possibilities for enriching the language of the dialogical framework with the means of CTT. However, though Chap. 10 deals to some extent with mathematical content and contains some brief remarks on empirical content, we shall here content ourselves with the more modest task of setting the basis for future, more thorough, developments on the issue.

## The Ancient Greek Roots of the Dialogical Turn and Its Renaissance

Before turning to the links between CTT and the dialogical framework, let us point out that Lorenzen's *dialogical turn* did not come out of the blue: Lorenzen was an admirer of Ernst Kapp's (1942) perspective on the dialectical origins of logic, and had a frequent and lively interaction with the philologist Kurt von Fritz. In fact, Lorenzen had a thorough and intimate knowledge of Ancient Greek mathematics and logic, even before he gathered the chair in Kiel in 1956, where he continued these kinds of studies then in contact with Oskar Becker (who influenced Lorenzen's appointment). In this context it might also well be that the inception of operative logic had a dialectical background that finally found its explicit expression in his *Logik und Agon* (1958).<sup>11</sup>

A striking witness of the Ancient Greek roots of the passage from the operative to the dialogical framework is *Ein formales Modell der Syllogistik des Aristoteles* (1964) by Lorenzen's student Kurt Ebbinghaus, where, after developing a remarkable proof-theoretical reconstruction of Aristotle's syllogistic in the style of *Operative Logik*, he discusses the advantages of a dialogical approach to Aristotle's notion of quantification—see Ebbinghaus (1964, pp. 57–58).<sup>12</sup>

In this context it is worth mentioning that nowadays history of mathematics is experiencing a revival in the studies linking the development of deductive proof in Ancient Greek mathematics with the dialogical practices of those days. Some of the most thorough studies on the subject are the ones of G. E. R. Lloyd (1996) and

<sup>11</sup>Kuno Lorenz conveyed this information to S. Rahman by a personal email.

<sup>12</sup>See (Crubellier, Marion, McConaughey, & Rahman, 2018) and (Rahman & Lion, 2018).

Reviel Netz (1999, 2005, 2009) who stress the importance of debates and oral dialogues for the emergence of classical mathematics in Ancient Greece. It seems like from the very start of mathematics the notion of proof was associated with the endeavour of explaining *why* the putative statement is true. Explaining *why* something is the case requires conceiving this explanation as directed towards a *stubborn interlocutor*, a point which does not hold only for the notion of proof in Ancient Greek mathematics—see (Fischer, 1989, p. 50).

Let us end this more historical section with the remark that the normative approach underlying the Dialogical Constructivism Program of Erlangen that emerged from the dialogical turn coupled explaining *why a putative statement is true* with the task of explaining *what the statement is good for*: according to the Erlangen-Programme the general notion of explaining is always conceived as *explaining to an audience what the purposes of an specific action (that give rise to the claim) are*—see Lorenzen (1969), Kamlah and Lorenzen (1972), Lorenzen and Schwemmer (1975).<sup>13</sup>

The general epistemological lesson behind Lorenzen’s bold proposal of a dialogical turn might be put in the following words: the dialogical turn is an invitation to think of actions involving scientific enquiry as interaction. It took a while until the scientific community picked up Lorenzen’s gauntlet, but as the most recent studies and projects in history and philosophy of logic, mathematics, foundations of computer sciences, linguistics, and epistemology point out, the time seems ripe now for the development of such a perspective.<sup>14</sup>

## 1.2 Linking Dialogues and Constructive Type Theory

### 1.2.1 Equality and the Socratic Rule

One of the main tenets of the present study is that a direct way to implement the Constructive Type Theoretical (CTT) approach within the dialogical framework is to focus on the CTT notion of judgemental equality.

In CTT, every category needs to be associated with a criterion of identity (see Chap. 2, written by Ansten Klev). More precisely, there are two basic forms of categorical judgement in CTT:

- (i)  $a : C$
- (ii)  $a = b : C$

<sup>13</sup>For a brief presentation of the philosophical tenets of Dialogical Constructivism see Sect. 11.7.

<sup>14</sup>See, among others, (Fischer, 1989), (Sellars, 1991), (Brandom, 1997), (Girard, 1999), (Heinzmann, 2006), (Ginzburg, 2012), (Lecomte, 2011), (Lecomte & Quatrini, 2010), (Paseau, 2011), (Peregrin, 2014), (Duthil Novaes 2015).

The first is read “ $a$  is an object of the category  $C$ ”, and the second, the judgemental or definitional equality, is read “ $a$  and  $b$  are identical objects of the category  $C$ ”. We thus require that any category  $C$  occurring in a judgement of CTT be associated with a

- *criterion of application*, which tells us what a  $C$  is; the fact that  $a$  meets this criterion is precisely what is expressed in  $a : C$ ; and a
- *criterion of identity*, which tells us what it is for  $a$  and  $b$  to be identical  $C$ ’s; the fact that  $a$  and  $b$  together meet this criterion is precisely what is expressed in  $a = b : C$ .

In the dialogical framework, on the other hand, equality involves plays in which players explicitly expose in the object language<sup>15</sup> the reasons they have for stating judgements. More precisely, as mentioned in the preface, definitional equality is implemented at the play level by means of the Socratic rule.

The Socratic rule is one of the most salient characteristics of dialogical logic. As discussed by Marion & Rückert (2015), it can be traced back to Aristotle’s reconstruction of the Platonic dialectics. A purely argumentative point of view can be defined within dialectics as refraining from calling on some authority beyond what has actually been brought forward during the current argumentative interaction (following the suitable rules determined by the game). Thus, when an elementary statement is challenged, the challenge can be answered only by invoking the challenger’s own concessions (or his own constructions). In such a context, the Socratic rule can be understood in the following way, when a player plays an elementary statement:<sup>16</sup>

*My reasons for stating this proposition you are now challenging are exactly the same as the ones you brought forward when you yourself stated that very same proposition.*

*In this fashion the Socratic rule provides for equality, but through interaction: equality is built within an argumentative play by copying exactly the same reasons for a proposition as what the other player has already provided.* Statements of definitional equality have thus emerged in a dialogical perspective, in particular reflexivity statements such as

$$p = p : A$$

which express the fact that if the Opponent states the elementary proposition  $A$ , then the Proponent can do the same, that is, play the same move and do it on the same grounds which provide the meaning and justification of  $A$ , namely  $p$ .

<sup>15</sup>This is the main feature of dialogues for immanent reasoning, the dialogical framework which incorporates features of CTT. For a presentation of this framework, see Chaps. 6 and 7. The Socratic rule is the equivalent in immanent reasoning of the Copy-cat rule in the standard dialogical framework. For a presentation of the standard framework, see Chaps. 3, 4, and 5.

<sup>16</sup>See (Rahman, Clerbout, & Keiff, 2009) and Rahman and Keiff (2010).

In order to introduce in the object language of the dialogical framework (dialogues for immanent reasoning) definitional identities at the play level, we must extend the language of a dialogical game with statements of the form

$$p : A$$

where on the right-hand side of the colon is the proposition  $A$ , and on the left-hand side is the *local reason* brought forward to back the proposition *during a play* (see Chap. 6). The local reason is therefore *local* if the force of the statement is limited to the level of plays. But when the assertion  $p : A$  is backed by a *winning strategy*, the judgement asserted draws its justification precisely from that strategy, thus endowing  $p$  with the status of a *strategic reason* (see Sect. 7.7).

Thus *reasons* backing a statement are manifest at the object language level, and are internal to the development of a play, which is why we have named this dialogical framework incorporating CTT features making these reasons explicit, *dialogues for immanent reasoning*.

The notion of *reason* (local and strategic) shows how we link the dialogical framework to CTT, but also how we can preserve the flexibility of the dialogical framework and bring out its full potential, ranging from material dialogues (at the play level) to the equivalent of the CTT demonstrations (at the strategy level) and all that which comes in-between. Immanent reasoning and equality in action are in this sense not exclusively at one level, but are embedded in the whole framework through the constitutive role of the Socratic rule.

### 1.2.2 Local Reasons and Content: The Socratic Rule Within Material Dialogues

Local reasons are fundamental to dialogues for immanent reasoning as they also contribute to *material dialogues* for elementary propositions (see Chap. 10). Informally, the idea is that if the Proponent is entitled to his statement on the elementary proposition  $A$ , it is because he is ready to defend  $A$  by giving a *reason* in favor of that statement. The Proponent can find such a local reason backing  $A$  in a process governed by the Socratic rule which spells out the precise forms of the local reason required by the *content* of  $A$ . The appropriate local reason will thus be governed by the Socratic rule (which ensures, by preventing the Proponent to provide his own grounds for what he says, that the grounds for stating an elementary proposition are taken from the play itself, that is, it ensures the reasons are immanent to the play), but this rule will have to be adapted to each individual content brought forward, bringing us to *material dialogues*, and should be contrasted with the development of *formal dialogues* in which the Socratic rule allows the Proponent to replicate an elementary proposition  $A$  stated by the Opponent, but also to replicate the local reasons that the Opponent brought forward when stating  $A$ , and this independently of checking what

these local reasons are: in purely formal dialogues, if the Opponent states for instance that 2 is odd, the Proponent can state this too on the sole grounds that the Opponent herself stated it and provided some reason for it, whatever this reason be.

But in material dialogues, if the Proponent asserts for example: “1 is an uneven number,” the Opponent would be entitled to request of the Proponent a natural number  $n$  such that  $1 = 2 \cdot n + 1$ . The local reason in this case would be  $1 = 2 \cdot 0 + 1$ , which is a reason specific to that particular statement the Opponent challenged.

In these dialogues, the Socratic rule determines the canonical elements and the definitions (the definitional equalities) specific to each of the elementary expressions in a play. This yields *material truth*.

Stating the *material truth* of a proposition requires exhibiting a local reason *specific* to the content of that proposition.

The origins of the normative approach to meaning can be found in this aspect of the dialogical framework: *meaning as use* should be understood as *the use is spelled out by a rule of dialogical interaction* which applies to the meaning of the logical constants, but also to the meaning of the elementary propositions. Strictly speaking, the meaning of each elementary expression requires a specific rule that determines its proper content and distinguishes it from other elementary propositions.<sup>17</sup> Material dialogues in this perspective are not only a matter of putting back normativity in logic, they also deal with the important matter of elaborating a contentful language.

### 1.2.3 *Dialogues for Immanent Reasoning as Games of Why and How*

The present study aims at showing that, if we follow Lorenzen and Lorenz’s advice of looking at mathematical operations as interaction, then definitional equality can be considered as exposing the dialogical intertwining of *entitlements* and *duties*. In this perspective, the standard monological presentation of these rules for both definitional and predicative equality implicitly encodes an underlying process, a process in which the Proponent “copies” some of the Opponent’s choices, thus providing its dialogical and normative roots.

We shall in this fashion rally to some extent to Robert Brandom’s insight<sup>18</sup> that conceptual meaning is entirely constituted by the way judgements are inserted into *games of giving and asking for reasons*, the touchstone of inferential pragmatism. Our task now lies in describing, in the context of these *games of giving and asking for reasons*, the moves on the ontological level grounding statements of equality

<sup>17</sup>Jaroslav Peregrin (2014, pp. 3–5) calls the notion of *use* understood as *following a rule* “role”. *Role* distinguish linguistic uses from other uses such as using a hammer.

<sup>18</sup>See for instance (Brandom, 1994). To some extent only, for it seems like Brandom starts from the strategy level rather than from the play level as we do.

(definitional equality), and on the propositional level grounding statements of identity (the dyadic-predicate of standard first-order logic). This is necessary in order to have games of *Why* and *How*.<sup>19</sup>

The emergence of concepts, we claim, are not only games of giving and asking for reasons (games involving *Why*-questions), they are also games that include moves establishing *how it is that the reason brought forward accomplishes the explicative task*. Immanent reasoning are dialogical games of *Why* and *How*.

We call our dialogues involving rational argumentation *dialogues for immanent reasoning* precisely because *reasons* backing a statement, now *explicit* denizens of the object language of plays, are *internal* to the development of the dialogical interaction itself.

### 1.3 A Basic Overview of the Book

One of the important lessons of the Constructive Type Theory approach to meaning is that equality is at the center of a constructivist project of types. Indeed, it has been stressed that the constructivist parallel to Quine's (1969, p. 23) notorious "*no entity without identity*" is

- No entity without a type
- No type without a criterion of identity

Definitional equality is central to the constitution of a type. Moreover, in the context of logic, definitional equality makes the coordination of analytic and synthetic steps explicit. So, if we are looking at ways of linking the normativity of dialogical logic with the normativity of CTT, it becomes apparent that we should try to provide an answer to the question of how the criterion of identity of a type is manifested in the dialogical framework—this is what the book is about.

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<sup>19</sup>As discussed in Sect. 10.5, Brandom's approach only has the propositional level (i.e. his framework does not include the ontological level of the *local reasons* relevant for the backing of the proposition involved in the judgement). Perhaps because he fears that such a move would amount to incorporating into the framework an *authority* which would be *external* to the games that determine concepts. As far as we understand it, this is a serious limitation of Brandom's approach since it fails to distinguish between the notations, or written forms, concerning the ontological level, and those concerning the propositional level: the present book, we hope, shows how to make the ontological level immanent to the dialogical process of reasoning. This suggests that the dialogical approach to CTT offers a way to integrate within one epistemological framework the two conflicting readings of Willfried Sellars' (1991, pp. 129–194) notion of *space of reasons* brought forward by John McDowell (2009, pp. 221–238) on the one side, who insists in distinguishing world-direct thought and knowledge gathered by inference, and by Robert Brandom (1997) on the other side, who interprets Sellars' work in a more radical anti-empiricist manner. The point is not only that we can deploy the CTT-distinction between reason as a premise and reason as the piece of evidence justifying a proposition, but it is also that the dialogical framework allows distinguishing between the objective justification (strategy) level targeted by Brandom (1997, p. 129) and the subjective (play) level stressed by McDowell—see also (Rahman, 2017).

The purpose of the book is therefore rather technical, though it has deep philosophical roots in what argumentation and reasoning are. Perhaps one way to condense our philosophical perspective on identity is that it has been developed in the general epistemological framework according to which

*argumentation is, all in all, nothing more—but nothing less—than a collaborative enquiry into the ways of constructing the symmetries grounding rationality within inquisitive interaction.*<sup>20</sup> *By building these symmetries we provide meaning to our actions, a meaning deployed in our actions' internal coordination with the actions of others (interaction).*

In order to allow readers to follow the technical aspects of this book, we have divided it into progressive sections, with examples and exercises.

The next section (Chap. 2) provides an elementary introduction to Constructive Type Theory (CTT), original in its equal emphasis on basic ideas and finer technical details.

The third chapter introduces essential notions for the dialogical framework and provides a basic and step by step approach to dialogues.

The fourth and fifth chapters aim at more advanced readers, who are either already familiar with the dialogical framework or well versed into logic; they respectively deal with the two fundamental levels characteristic of dialogues: the play level and the strategy level.

The reader should by then be fully equipped for the following sections, which are the core of this book and deal precisely with the problem at issue, that is, with immanent reasoning and equality in action. Thus the sixth chapter introduces local reasons in the dialogical framework, a crucial step for immanent reasoning; the seventh chapter presents again the local reasons but from a more technical side, deals with the strategy level in dialogues for immanent reasoning, and introduces a key notion: strategic reasons, the dialogical counterpart to CTT proof-objects.

The eighth chapter illustrates some of the imports of the constructive perspective in general and of dialogues for immanent reasoning in particular through the case of the Axiom of Choice.

The ninth chapter provides an algorithm allowing to go from dialogical strategies to CTT demonstrations, and reversewise.

The tenth chapter touches on the less studied material dialogues, that is, dialogues with rules for local and global meaning that are not restricted to the rules of logical constants. We study the case of propositional identity, the set **Bool**, Boolean operations, and finite sets.

The final chapter will be our conclusion, which contains some philosophical remarks on dialogical constructivism and suggests some responses to standard objections to the dialogical framework.

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<sup>20</sup>For more details on symmetry in the dialogical framework, see Sect. 4.3.



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