Fred Espen Benth
Almut E. D. Veraart *Editors*

Quantitative Energy Finance

Recent Trends and Developments

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Preface

The European power systems undergo a huge transition towards renewable generation, having a significant impact on risk management and operations in the markets for electricity. Gas- and coal-fired power plants are one of the major sources of carbon emissions, and have to be substituted to reach the net-zero goals of Europe. This rapid transition creates new and challenging problems within quantitative energy finance, some of which we address in this volume.

In the book *Quantitative Energy Finance – Modeling, Pricing, and Hedging in Energy and Commodities Markets* (F. E. Benth, V. A. Kholodnyi and P. Laurence (eds.), Springer Verlag 2014), the focus was on bridging risk management tools from financial theory over to energy and commodity markets, with a particular view on power. Nearly a decade later, the markets for electricity in Europe have experienced a huge development, with closer integration between regions, fast development of renewable power and regulatory changes such as the EU taxonomy on sustainable finance. The markets after 2020 have also been hit by unprecedented highs and lows of electricity prices, explained by longer periods of little to no wind over Europe, a high degree of intermittency in the generation, and the cut in the import of Russian gas. In the future, climate change is predicted to further impact the power markets, with changing weather patterns leading to more frequent extreme weather such as heat waves, "Dunkelflaute" and cold spells. The electrification of society (transport, households, industry) leads, on the other hand, to an increased demand for power.

With the current volume *Quantitative Energy Finance – Recent Trends and Developments*, which is a stand-alone continuation of the book published in 2014, we have collected a set of scientific papers analysing important aspects and challenges that we see for the moment and on the way ahead towards a net-zero energy system.

We have grouped the papers according to three broad topics: The first group of articles is concerned with the *modelling of energy prices* taking recent changes in energy generation into account, followed by articles on the *energy transition*, and we conclude the book with a recent survey on the topic of *climate risk*.

We will now briefly summarise the main contributions of each chapter.

Modelling of Energy Prices

Estimation of the Number of Factors in a Multi-factorial Heath–Jarrow–Morton Model in Power Markets by Olivier Féron, and Pierre Gruet: This chapter advances calibration methods for multi-factorial Heath–Jarrow–Morton models in the context of power markets with a particular focus on determining the optimal number of Gaussian factors in the model. The authors calibrate the model jointly on both spot and futures prices using maximum-likelihood techniques combined with information criteria. In an empirical study of Belgian, French, and German power prices, they demonstrate close similarities between the three markets and the number of factors needed to model the prices well.

Hawkes Processes in Energy Markets: Modelling, Estimation and Derivatives Pricing by Riccardo Brignone, Luca Gonzato and Carlo Sgarra: In this chapter, the authors first review recent developments in using Hawkes processes to model energy prices and carry out derivatives pricing, including a description of exact simulation methods for Hawkes processes. Next, they propose a stylised new model for energy spot prices, which is built on a Hawkes process. Since this model is formulated under the historical probability measure, they furthermore establish a structure-preserving change of measure to also describe the corresponding risk-neutral dynamics of the spot prices needed for derivatives pricing. Using particle filtering techniques, the model can be estimated and an application to pricing exotic derivatives concludes this work.

Periodic Trawl Processes: Simulation, Statistical Inference and Applications in Energy Markets by Almut E. D. Veraart: This chapter introduces the new class of continuous-time periodic trawl processes, which can account for periodic behaviour in the serial correlation either in a short- or long-memory framework. It presents their probabilistic properties and establishes the asymptotic theory for (generalised) method of moments estimators for the model parameters and proposes efficient simulation schemes for such processes. The methodology is applied to electricity spot prices from the German electricity market.

Energy Transition

Fuelling the Energy Transition: The Effect of German Wind and PV Electricity Infeed on TTF Gas Prices by Christoph Halser and Florentina Paraschiv: In order to facilitate the energy transition, low-carbon and flexible balancing tools are needed to deal with the intermittency of renewable energy generation. Gas has been playing an important role in the current energy transition and hence the authors study the substitution effect between gas and renewable energies wind and PV. They carry out their analysis in the context of threshold regression models applied to recent daily Dutch natural gas prices. They find a negative marginal effect of the day-ahead wind and PV infeed forecasts on day-ahead natural gas prices and a positive association between the day-ahead gas price and CO2 prices, coal prices, heating demand and supplier concentration.

A Mean-Field Game Model of Electricity Market Dynamics by Alicia Bassiére, Roxana Dumitrescu and Peter Tankov: This chapter develops a mean-field game model for the long-term dynamics of electricity markets. This new model includes various refinements over existing models, such as that an arbitrary number of technologies with endogenous fuel prices can be considered, agents can both invest and divest, and various temporal aspects of the plant construction and age can be incorporated. The new model aims to describe the impacts of energy transition on electricity markets with a particular focus on the role gas plays, in the medium term, as a substitute for coal. The authors illustrate the properties of the new model through numerical computations and a stylised example.

PPA Investments of Minimal Variability by Fred Espen Benth: A power purchase agreement (PPA) is a long-term financial contract between an electricity generator and a customer. In this chapter, the focus is on a PPA, where one can virtually operate a solar or wind power park. Since renewable energy generation is highly volatile, such a PPA could be used as a spatial hedge, where production is spread out geographically. This chapter proposes a model for the capacity factors of solar and wind generation by a square-integrable random field in a Hilbert space with an associated covariance operator and analyses how the variability of a portfolio of power plants spread out over various spatial locations can be minimised. In a case study of a PPA of a portfolio of solar power plants in Germany, it is shown that the variability of the difference between solar power production and electricity demand can be reduced significantly by a spatial hedge.

Climate Risk

Climate Risk in Structural Credit Models by Alexander Blasberg and Rüdiger Kiesel: The book concludes with a timely survey on how the impact of climate risk on financial markets can be described by structural credit risk models. Physical and transition risks, often considered the key components of climate risk, can be captured by the classical Merton model and its extensions, and the authors carefully describe the advantages and shortcomings of the existing models and outline possible improvements.

All chapters have been refereed by peers, to whom we are grateful for their (anonymous) contribution to the scientific quality of this book.

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Part I
Modelling of Energy Prices

Estimation of the Number of Factors in a Multi-Factorial Heath-Jarrow-Morton Model in Power Markets

Olivier Féron and Pierre Gruet

Abstract We study the calibration of specific multi-factorial Heath-Jarrow-Morton models to power market prices, with a focus on the estimation of the optimal number of Gaussian factors. We describe a common statistical procedure based on likelihood maximisation and Akaike/Bayesian information criteria, in the case of a joint calibration on both spot and futures prices. We perform a detailed analysis on three national markets within Europe: Belgium, France, and Germany. The results show a lot of similarities among all the markets we consider, especially on the optimal number of factors and on the behaviour of the different factors.

1 Introduction

Electricity generation and supply have been widely liberalised in a large set of countries over the last decades. Although their precise organisation varies across places, power markets share a common structure linked to the specificities of electricity: it is not storable and therefore has to be produced exactly when it is consumed. For instance, in Western Europe, the *spot market* takes place everyday and allows one to define the amounts of electricity that will be produced (and consumed) during each of the hours in the next day, based on quite accurate forecasts of consumption needs and production capacities. However, as prices are very volatile on the spot market, utilities may casually want to avoid having their full production exposed to the spot price, and they can mitigate their financial exposure on the spot market by trading derived products on the *financial futures market*. On this futures market, standardised contracts can be exchanged continuously for the next weeks, months, quarters and years or seasons. Grasping the characteristics of the evolution of prices on the futures market is essential to be able to use it efficiently by computing relevant hedging strategies and risk indicators.

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We address the statistical estimation of a family of models for electricity prices and propose a methodology to select one of those models using information criteria. We consider Heath-Jarrow-Morton (HJM) models, introduced in [18] to represent the dynamics of the forward rates. In that work devoted to the term structure of interest rates, the forward rates processes were led by a sum of N Brownian motions and a drift. Being common to all maturities, this set of stochastic factors was driving the whole forward rate curve. Reference [18] focuses on the use of their model for valuing contingent claims, and some examples are given.

Using such models to represent electricity prices has been done by many authors. Reference [5] defined a 1- and a 3-factor HJM models to represent the dynamics of instantaneous delivery futures in power markets. As they acknowledged that the actually traded contracts are flow futures, meaning there is some delivery period, they used an approximation studied in [23] to derive a valuation formula for such flow futures contracts. In contrast, [7] applied HJM models to oil prices: they designed a methodology to search for the best number of factors by doing a Principal Component Analysis (PCA) on their data. Reference [25] followed the same approach on electricity prices, while accounting for the existence of the delivery period. Those last two articles let the volatility functions be totally unspecified: the PCA leads to nonparametric volatility functions. As they did, many authors have also looked for the best number of stochastic factors and for the shape of the volatility coefficients: keeping them simple ensures the models can be used operationally for risk management purposes and can lead to simple formulas for prices of derivatives. Reference [27] designed 1- and 2-factor models for the electricity spot price only. Reference [22] designed 1- to 4-factor models for electricity prices, accounting for the delivery period, and designed an extended Kalman filter to estimate the models on data from Nordpool market. They suggested to use 2- or 3-factor models, but had no mathematical criterion to argue. Their model incorporates a noise process, which features market imperfections. Reference [28] proposed a sum of two Ornstein-Uhlenbeck mean-reverting processes to represent gas futures prices, which they estimated on Henry Hub price data. Reference [10] studied the risk premia in power markets with a 3-factor model that they estimated using a two-step procedure. Reference [24] introduced a 2-factor model for electricity prices, which they calibrated on German market to implied volatilities. The same model was studied in [13] where the calibration results show instability of the parameters, depending on the data that are considered. Reference [1] discussed the use of HJM models for futures contracts in power markets. They explained the implications of various modelling choices from a practical viewpoint. Reference [11] proposed a 2-factor model similar to the one in [24]. Their model does not allow for delivery periods, but it can account for more commodities to be correlated. It is applied on oil prices data after many estimation methods are described. References [2, 26] recently proposed HJM-type additive models to jointly represent instantaneous and flow futures. The former article lists conditions so that such models do not allow for arbitrage. It gives examples of simple models suiting their frame. The latter article performs estimation in such an additive model with 2 factors, by minimising the difference between the theoretical and empirical covariations of processes.

Reference [14] performed the efficient estimation of a 2-factor model with stochastic volatility, working within a market model representing the dynamics of futures contracts with a delivery duration of one month. For a thorough introduction to factor models as well as a review of articles using them to model electricity prices, one may refer to [9].

In the context of interest rates, [3] estimated HJM models on Australian interest rates data. In their model, the volatility is a function of the level of the process. Reference [4] performed maximum likelihood estimation of a 1-factor model on American short term interest rates data. Reference [19] discussed the estimation of HJM models on German bond data by performing a PCA, and then by using nonlinear regression to estimate parametrically four specific models. Reference [20] worked on HJM models for interest rates dynamics where the volatility is an unspecified function of the rate level. They acknowledged that there are less Brownian motions than yield curves to be represented in their model, which implies stochastic singularity: some deterministic relationships between yield curves hold using the model, although they do not hold on real-world data.

In order to be able to select a relevant model, one of the main stakes is to represent the volatility structure of prices. Among all possible models, we have to find an equilibrium between a good quality of representation and a simple and parsimonious form. We are focusing on models in which the dynamics of prices is driven by a sum of correlated Brownian motions, with deterministic volatility coefficients which decrease exponentially as the remaining time to delivery increases. This class of models is very well known and used in practice for its tractability to deal with option pricing and hedging purposes. It encompasses the set of seminal commodity models [15, 27, 30], which are used for pricing derivatives, for example, in [8] and more recently in [12]. Our aim is not to find the best price model, but rather to find differences and similarities on different power markets by means of a quite simple class of HJM models. We focus our study on the number of needed Brownian motions as a function of the market and the data used in order to be parsimonious while reaching a good quality of representation. To this end, we compute the classical Akaike information Criterion (AIC) and Bayesian information criterion (BIC), and we also propose some additional indicators to help the user choose an efficient number of factors.

Selecting the appropriate number of stochastic components to describe electricity prices was also the focus of [16]. Starting with a Gaussian Ornstein-Uhlenbeck stochastic factor, they added Lévy-driven Ornstein-Uhlenbeck processes to the dynamics to model either positive or negative jumps until the predictive p-values of their models are satisfactory. Based on deseasonalised daily electricity spot prices without the weekends from the UK and Europe on various historical periods, they found that (depending on the datasets) one or two jump components give a satisfactory quality of representation given their metrics. Here we shall only consider Gaussian processes but the estimation will be performed on spot and futures contracts jointly.

The rest of the chapter is organised as follows. In Sect. 2 we present the model and recall the corresponding dynamics of the prices for spot and futures contracts with delivery periods. The estimation procedure is described in Sect. 3. In Sect. 4 we precisely describe the data used for the estimation, and we show and analyse the estimation results. Section 5 concludes this work and outlines some future research directions.

2 Model Description

In this section we introduce the model for electricity price futures and we derive the equation for spot prices. In the sequel we use the following notation:

- $F_t(T)$ denotes the unitary power futures price at date t for the delivery of one megawatt-hour (hereafter MWh) of electricity at date T. Such a unitary futures is not traded on the markets, but is used as a modelling brick to write the spot price and the prices of quoted futures, see [9] for a thorough discussion on this approach;
- $S_t = F_t(t)$ is the power spot price;
- $F_t(T, \theta)$ denotes the power futures price quoted at t, delivering 1 MWh during all the hours between times T and $T + \theta$, where θ is the length, in years, of the delivery period.

Let $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ be a filtered probability space, and let us assume the absence of arbitrage. There exists a unique risk-neutral measure $\mathbf{Q}$ equivalent to $\mathbf{P}$, and under this risk-neutral measure we consider the classical Heath-Jarrow-Morton [18] model written on the unitary futures price:

$$\frac{dF_t(T)}{F_t(T)} = \sum_{k=1}^{N} e^{-\alpha_k(T-t)}\sigma_k dW_t^k \ , \tag{1}$$

where $N \geq 1$ is the number of stochastic factors, $(W^k)_{k=1,\dots,N}$ is a N-dimensional $\mathbf{F}$-adapted $\mathbf{Q}$-Brownian motion, of which components k and k' have correlation $\rho_{k,k'}$, $\sigma_k > 0$ for $1 \leq k \leq N$, and $0 < \alpha_1 < \cdots < \alpha_N$ in order to guarantee identifiability of the model.

By integrating Eq. (1) and letting $T = t$, we deduce the expression of the spot price S_t as a function of $F_{t_0}(t)$, for $t_0 \leq t$:

$$\begin{aligned} S_t = F_{t_0}(t) \exp\Bigg\{ &-\frac{1}{2}\sum_{k=1}^{N}\sum_{k'=1}^{N} \rho_{k,k'}\sigma_k\sigma_{k'}\frac{1-e^{-(\alpha_k+\alpha_{k'})(t-t_0)}}{\alpha_k+\alpha_{k'}} \\ &+ \sum_{k=1}^{N}\int_{t_0}^{t}\sigma_k e^{-\alpha_k(t-s)}dW_s^k \Bigg\} \ . \end{aligned} \tag{2}$$

Applying Itô formula and assuming that F_{t_0} is differentiable, we get the dynamics of the spot price, namely

$$\frac{dS_t}{S_t} = \left(\frac{F'_{t_0}(t)}{F_{t_0}(t)} + \frac{1}{2} \sum_{k=1}^{n} \sum_{k'=1}^{N} \sigma_k \sigma_{k'} \rho_{k,k'} \left(1 - e^{-(\alpha_k + \alpha_{k'})(t-t_0)}\right) \right) dt + \sum_{k=1}^{N} dZ_t^k ,$$

where we introduced the auxiliary processes Z^k, $k = 1, \ldots, N$, defined by

$$\begin{aligned} dZ_t^k &= -\alpha_k Z_t^k dt + \sigma_k dW_t^k , \\ Z_{t_0}^k &= 0 , \end{aligned} \tag{3}$$

for $1 \leq k \leq n$. It is worth emphasising that the dynamics of the spot price can be written as being led by a sum of Ornstein-Uhlenbeck processes.

Concerning the futures prices, as in [13], we can consider the no-arbitrage (discrete) relationship between futures contract prices and unitary futures prices in the form

$$F_t(T, \theta) = \frac{h}{\theta} \sum_{i=0}^{\theta/h-1} F_t(T + ih) ,$$

where h is a the timestep (1 hour or 1 day for example) considered in the discretization of the forward curve. Combining this equation with Eq. (1), we can deduce the dynamics of the futures contract prices:

$$\begin{aligned} \frac{dF_t(T,\theta)}{F_t(T,\theta)} &= \frac{1}{F_t(T,\theta)} \frac{h}{\theta} \sum_{i=0}^{\theta/h-1} F_t(T+ih) \sum_{k=1}^{N} e^{-\alpha_k(T+ih-t)} \sigma_k dW_t^k \\ &= \sum_{k=1}^{N} e^{-\alpha_k(T-t)} \sigma_k \left(\frac{h}{\theta} \sum_{i=0}^{\theta/h-1} \frac{F_t(T+ih)}{F_t(T,\theta)} e^{-\alpha_k ih} \right) dW_t^k . \end{aligned} \tag{4}$$

3 Estimation

In this section we describe the estimation methodology based on Kalman filtering and the maximum likelihood principle, and we explain how we compute the classical AIC and BIC in order to study the optimal number of factors. The likelihood function is obtained from a state-space equation system and computed via a Kalman filter: the likelihood function is the one of the residuals of the filter, which are multivariate Gaussian at each time step and are independent from one time step to the other. Also, as already described in [17], we will introduce a Gaussian error model to face the stochastic singularity (see [20]) when the number of model factors is lower than the number of observed futures contracts.

In the sequel we consider $n+1$ quotation dates $t_0, \ldots, t_n$ and we use the notation $\Delta_{t_i} = t_i - t_{i-1}$ and $\Delta_i^n X = X_{t_i} - X_{t_{i-1}}$ for any process X. Thus the index n stands for the number of updates in the Kalman filtering step, and thus for the number of Kalman residuals that will be computed.

We will assume that the historical measure $\mathbf{P}$ and the risk-neutral measure $\mathbf{Q}$ coincide, so that the $\mathbf{Q}$-Brownian motions are also $\mathbf{P}$-Brownian motions and the dynamics (1) also holds under $\mathbf{P}$. Our motivation to do so is twofold:

- Assuming the dynamics (1) holds under $\mathbf{Q}$, we could consider a dynamics with a drift term under $\mathbf{P}$, for example

$$\frac{dF_t(T)}{F_t(T)} = \sum_{k=1}^{N} b_k e^{-\alpha_k(T-t)}dt + \sum_{k=1}^{N} e^{-\alpha_k(T-t)}\sigma_k d\widetilde{W}_t^k ,$$

  where the $\widetilde{W}^k$ are $\mathbf{P}$-Brownian motions and $b_1, \ldots, b_N$ are real numbers. There is no technical difficulty in making inference about those N real numbers along with the volatility parameters by following the methodology described hereafter. Still this increases the dimension of the optimization problem while our goal is to discuss the volatility structure;
- As [10] reported in 2006, there is empirical evidence that the absolute values of risk premia decrease as markets become more mature and attract speculators.

3.1 Distributions of the Changes in Futures and Spot Prices

We consider the approximation of the futures prices dynamics, as in [13], stating that the shaping factors $\frac{F_t(T+ih)}{F_t(T,\theta)}$ are all equal to 1. This assumption boils down to asserting the contribution of each hour in a given delivery period to the price of the whole delivery is the same, and then that, for each delivery period $[T, T+\theta]$, all prices $F_t(s)$ for $s \in [T, T+\theta]$, are identical and equal to the observed futures price $F_t(T, \theta)$. In order to avoid inconsistency, it is necessary to consider futures with disjoint delivery periods: for any couple $(F_t(T_1, \theta_1), F_t(T_2, \theta_2))$ of futures contracts considered in the sequel, we have $[T_1; T_1+\theta_1] \cap [T_2; T_2+\theta_2] = \emptyset$. In practice, this means that a preprocess (described hereafter in Sect. 4.1.2) is needed to remove any overlap in the futures' delivery periods. With these assumptions, one can compute the dynamics (4) as

$$\frac{dF_t(T,\theta)}{F_t(T,\theta)} = \sum_{k=1}^{N} e^{-\alpha_k(T-t)}\sigma_k \psi_h(\alpha_k, \theta) dW_t^k ,$$

where $\psi_h(\alpha, \theta) = \frac{h}{\theta}\frac{1-e^{-\alpha\theta}}{1-e^{-\alpha h}}$.

By denoting $X_t^{T,\theta} = \log(F_t(T,\theta))$ and applying Itô's lemma we get

$$\Delta_i^n X^{T,\theta} = \sum_{k=1}^{N} \int_{t_{i-1}}^{t_i} \psi_h(\alpha_k,\theta) e^{-\alpha_k(T-t)} \sigma_k dW_t^k \\ - \frac{1}{2} \sum_{k=1}^{N} \sum_{k'=1}^{N} \int_{t_{i-1}}^{t_i} e^{-(\alpha_k+\alpha_{k'})(T-t)} \rho_{k,k'} \sigma_k \sigma_{k'} \psi_h(\alpha_k,\theta) \psi_h(\alpha_{k'},\theta) dt \ . \tag{5}$$

At each time t_i we assume observing L_i prices of futures contracts[1] which are denoted by $F_{t_i}(T_\ell,\theta_\ell)$, for $\ell = 1,\ldots,L_i$. When $L_i > N$, the model presents a stochastic singularity (already studied in [20] and in [17] in the case of a two-factor model applied to electricity prices). In order to face this problem we introduce a model error term and assume observing noisy returns. Precisely, we do not observe the increments $\Delta_i^n X^{T_\ell,\theta_\ell}$ of the price process, but instead we observe

$$\Delta_i^n Y^{T_\ell,\theta_\ell} = \Delta_i^n X^{T_\ell,\theta_\ell} + \varepsilon_i^{T_\ell,\theta_\ell} \ ,$$

where $\varepsilon_i^{T_\ell,\theta_\ell}$ are identically distributed according to a Gaussian distribution $\mathcal{N}(0,\upsilon^2)$, where υ^2 is unknown. Moreover, for all $i = 1,\ldots,n$, the random variables $\varepsilon_i^{T_\ell,\theta_\ell}$ and $\varepsilon_i^{T_{\ell'},\theta_{\ell'}}$ are independent for $1 \le \ell < \ell' \le L_i$, as well as the random variables $\varepsilon_i^{T_\ell,\theta_\ell}$ and $\varepsilon_j^{T_{\ell'},\theta_{\ell'}}$ for $1 \le i < j \le n$ and $1 \le \ell \le L_i$, $1 \le \ell' \le L_j$.

Therefore, the vector $\left(\Delta_i^n Y^{T_1,\theta_1} \cdots \Delta_i^n Y^{T_{L_i},\theta_{L_i}}\right)'$ which stacks up the observed noise-perturbed increments at time step i is Gaussian $\mathcal{N}(\mathbf{M_i},\mathbf{\Sigma_i})$ with $\mathbf{M_i} = (M_i^\ell)_{1\le\ell\le L_i}$, $\mathbf{\Sigma_i} = (\Sigma_i^{\ell\ell'})_{1\le\ell,\ell'\le L_i}$ and:

$$M_i^\ell = \mathbf{E}\left(\Delta_i^n Y^{T_\ell,\theta_\ell}\right) \\ = -\frac{1}{2} \sum_{k=1}^{N} \sum_{k'=1}^{N} \Bigg(\rho_{k,k'} \sigma_k \sigma_{k'} \psi_h(\alpha_k,\theta_\ell) \psi_h(\alpha_{k'},\theta_\ell) \\ \times e^{-(\alpha_k+\alpha_{k'})(T_\ell - t_i)} \frac{1-e^{-(\alpha_k+\alpha_{k'})\Delta_{t_i}}}{\alpha_k+\alpha_{k'}} \Bigg) \tag{6}$$

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[1] In practice the number of observed prices may vary, even after removing the redundant products (e.g. a quarterly contract when all the corresponding monthly contracts are observed), see Sect. 4.1.

and

$$\begin{aligned}\Sigma_i^{\ell\ell'} =& \mathbf{C}ov\left(\Delta_i^n Y^{T_\ell,\theta_\ell}, \Delta_i^n Y^{T_{\ell'},\theta_{\ell'}}\right)\\ =& v^2 \mathbf{1}_{\ell=\ell'} + \sum_{k=1}^N \sum_{k'=1}^N \Bigg(\rho_{k,k'}\sigma_k\sigma_{k'}\psi_h(\alpha_k,\theta_\ell)\psi_h(\alpha_{k'},\theta_{\ell'})\\ & \times e^{-\alpha_k(T_\ell - t_i)} e^{-\alpha_{k'}(T_{\ell'}-t_i)} \frac{1-e^{-(\alpha_k+\alpha_{k'})\Delta_{t_i}}}{\alpha_k+\alpha_{k'}}\Bigg),\end{aligned}$$

where $\mathbf{1}_{\ell=\ell'} = 1$ if $\ell = \ell'$ and 0 otherwise.

Let us introduce the auxiliary processes Z^k, for $k = 1, \ldots, N$, defined by $Z_{t_0}^k = 0$ and

$$dZ_t^k = -\alpha_k Z_t^k dt + \sigma_k dW_t^k .$$

By integrating the above dynamics between dates t_{i-1} and t_i, we have

$$Z_{t_i}^k = Z_{t_{i-1}}^k e^{-\alpha_k \Delta_{t_i}} + \int_{t_{i-1}}^{t_i} \sigma_k e^{-\alpha_k(t_i - t)} dW_t^k$$

so that we can rewrite Eq. (5) with the auxiliary processes Z^k as:

$$\begin{aligned}\Delta_i^n X^{T,\theta} =& \sum_{k=1}^N \psi_h(\alpha_k,\theta) e^{-\alpha_k(T-t_i)}\left(Z_{t_i}^k - Z_{t_{i-1}}^k e^{-\alpha_k\Delta_{t_i}}\right)\\ & - \frac{1}{2}\sum_{k=1}^N\sum_{k'=1}^N \Bigg(\rho_{k,k'}\sigma_k\sigma_{k'}\psi_h(\alpha_k,\theta)\psi_h(\alpha_{k'},\theta)\\ & \times e^{-(\alpha_k+\alpha_{k'})(T-t_i)} \frac{1-e^{-(\alpha_k+\alpha_{k'})\Delta_{t_i}}}{\alpha_k+\alpha_{k'}}\Bigg).\end{aligned} \tag{7}$$

Concerning the spot price, we use expression (2) and we write $\widetilde{S}_t = \frac{S_t}{F_{t_0}(t)}$ for the seasonality adjusted spot price in order to obtain

$$\begin{aligned}\Delta_i^n X =& \log \widetilde{S}_{t_i} - \log \widetilde{S}_{t_{i-1}}\\ =& \sum_{k=1}^N \left(Z_{t_i}^k - Z_{t_{i-1}}^k\right)\\ & - \frac{1}{2}\sum_{k=1}^N\sum_{k'=1}^N \rho_{k,k'}\sigma_k\sigma_{k'} e^{-(\alpha_k+\alpha_{k'})(t_{i-1}-t_0)} \frac{1-e^{-(\alpha_k+\alpha_{k'})\Delta_{t_i}}}{\alpha_k+\alpha_{k'}}.\end{aligned} \tag{8}$$

As for the dynamics of the futures prices, we consider a model (or measurement) error in the observed spot prices. We assume observing $\Delta_i^n Y$ defined as

$$\Delta_i^n Y = \Delta_i^n X + \varepsilon_i \ ,$$

where the random variable ε_i is distributed according to a Gaussian distribution $\mathcal{N}(0, v^2)$ and is independent of all random variables $\varepsilon_j^{T_\ell,\theta_\ell}$, $j = 1, \ldots, n$, $\ell = 1, \ldots, L_j$. We may notice that $\Delta_i^n X$ cannot be considered as a price "return" because the underlying is the spot price, which stands for a different delivery period each day. We propose to consider this element in the calibration process in order to work with differences in price logarithms for all observed (futures and spot) prices.

3.2 State-Space System of Equations

Having derived the distributions of price changes at each time step in the previous subsection, the multi-factor model can be written in a state-space model formulation: let us denote

$$\Delta_i^n \mathbf{Y} = \begin{pmatrix} \Delta_i^n Y^{T_1,\theta_1} \\ \vdots \\ \Delta_i^n Y^{T_{L_i},\theta_{L_i}} \\ \Delta_i^n Y \end{pmatrix} , \qquad \mathbf{Z}_i = \begin{pmatrix} Z_{t_i}^1 \\ Z_{t_{i-1}}^1 \\ \vdots \\ Z_{t_i}^N \\ Z_{t_{i-1}}^N \end{pmatrix} .$$

The multi-factor model can be written as follows:

$$\begin{aligned} \Delta_i^n \mathbf{Y} &= \mathbf{M}_i + \mathbf{F}_i \mathbf{Z}_i + \boldsymbol{\varepsilon}_i \ , \\ \mathbf{Z}_i &= \mathbf{A}_i \mathbf{Z}_{i-1} + \boldsymbol{\eta}_i \ , \end{aligned} \tag{9}$$

with elements $\mathbf{M}_i$, $\mathbf{F}_i$, $\mathbf{A}_i$ and the covariance matrices of $\boldsymbol{\varepsilon}_i$ and $\boldsymbol{\eta}_i$ given in the upcoming subsections, in which we describe the components of the state-space system (9).

3.2.1 Elements of the Space Equation

$\Delta_i^n \mathbf{Y}$ is a vector of size $L_i + 1$, corresponding to the number of observed futures returns at date t_i and the deseasonalized spot price change. The first L_i components correspond to the futures prices returns and the last component corresponds to the spot. The mean vector $\mathbf{M}_i$ then has its L_i first components defined by Eq. (6) and its last component defined, accordingly, by the last term in Eq. (7).

Concerning the matrix $\mathbf{F}_i$, we propose to write it as the stack of two different linear forms $\mathbf{F}_i = ((\mathbf{F}_i^f)', (\mathbf{F}_i^s)')'$ corresponding to the futures contracts and the spot, respectively.

Using Eq. (7) we have

$$\mathbf{F}_i^f = G_i^f H_i^f$$

with G_i^f a $(L_i \times N)$ matrix:

$$G_i^f = \begin{bmatrix} \psi_h(\alpha_1, \theta_1)e^{-\alpha_1(T_1-t_i)} & \dots & \psi_h(\alpha_N, \theta_1)e^{-\alpha_N(T_1-t_i)} \\ \vdots & \dots & \vdots \\ \psi_h(\alpha_1, \theta_{L_i})e^{-\alpha_1(T_{L_i}-t_i)} & \dots & \psi_h(\alpha_N, \theta_{L_i})e^{-\alpha_N(T_{L_i}-t_i)} \end{bmatrix}$$

and H_i^f a $(N \times 2N)$ matrix:

$$H_i^f = \begin{bmatrix} 1 & -e^{-\alpha_1 \Delta t_i} & 0 & \dots & \dots\dots & & 0 \\ 0 & 0 & 1 & -e^{-\alpha_2 \Delta t_i} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 & \dots\dots & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \dots & \dots & \dots & 0 & 1 & -e^{-\alpha_N \Delta t_i} \end{bmatrix}.$$

The matrix H_i^f may not depend on time if the time step Δt_i is constant. However, G_i^f depends on time because of the maturity terms $T_\ell - t_i$.

Concerning the part $\mathbf{F}_i^s$ dedicated to the spot prices, we can use the same decomposition using Eq. (8):

$$\mathbf{F}_i^s = G_i^s H_i^s$$

with $G_i^s = \mathbf{1}'_N$ the transpose of a N-dimensional vector composed of ones, and H_i^s a $(N \times 2N)$ matrix:

$$H_i^s = \begin{bmatrix} 1 & -1 & 0 & \dots\dots\dots & & & 0 \\ 0 & 0 & 1 & -1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 & \dots\dots & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \dots\dots\dots & & & 0 & 1 & -1 \end{bmatrix}.$$

In the calibration tests, as we explained above, we assume that the variance of the model errors is identical for all observed contracts, i.e.

$$\mathbf{Q}_{\varepsilon_i} = v^2 \mathbf{I}_{L_i+1}$$

where $\mathbf{I}_m$ is the $m \times m$ identity matrix.

3.2.2 Elements of the State Equation

Using the solution of the Ornstein-Uhlenbeck processes on the factors Z^k defined in Eq. (3), we get the block-diagonal matrix

$$\mathbf{A}_i = \begin{bmatrix} e^{-\alpha_1 \Delta t_i} & 0 & \dots & \dots\dots & \dots & 0 \\ 1 & 0 & \ddots & \dots\dots & \dots & 0 \\ \vdots & \vdots & e^{-\alpha_2 \Delta t_i} & 0 \dots & \dots & \vdots \\ \vdots & \vdots & 1 & 0 \dots & \dots & \vdots \\ \vdots & \vdots & \vdots & \ddots \ddots & \dots & \vdots \\ 0 & \dots & \dots & \dots \; 0 & e^{-\alpha_N \Delta t_i} & 0 \\ 0 & \dots & \dots & \dots\dots & 1 & 0 \end{bmatrix}.$$

If the time step Δt_i is constant, then the matrix $\mathbf{A}_i = \mathbf{A}$ is also constant.

The $(2N \times 2N)$ covariance matrix $\mathbf{Q}_{\eta_i}$ is also deduced from the dynamics of the Ornstein-Uhlenbeck processes, and it is made from 2×2 blocks of which only the upper-left component is not zero. Precisely, for $1 \leq k, k' \leq N$,

$$\begin{aligned} \mathbf{Q}_{\eta_i}^{2k-1,2k'-1} &= \rho_{k,k'}\sigma_k\sigma_{k'} \frac{1 - e^{-(\alpha_k+\alpha_{k'})\Delta_{t_i}}}{\alpha_k + \alpha_{k'}} \,, \\ \mathbf{Q}_{\eta_i}^{2k-1,2k'} = \mathbf{Q}_{\eta_i}^{2k,2k'-1} = \mathbf{Q}_{\eta_i}^{2k,2k'} &= 0 \,. \end{aligned} \qquad (10)$$

3.3 *Implementation of the Kalman Filter and Minimization Algorithm*

Now we explain how the Kalman filter is implemented and how the log-likelihood of the Kalman residuals is computed. Kalman filtering is named after Kalman [21] and is aimed at addressing problems in which one tries to get information about some state process that is shaded within noisy measurements. This is done by making, at each time step, some prediction of the next state and then updating the internal variables of the filter by using the comparison of this prediction to the realized state as a feedback. Namely, in the state-space equation (9), $\mathbf{Z}_i$ is the hidden state at step i and it is driven by the evolution of the stochastic factors on which one is willing to make inference. $\Delta_i^n \mathbf{Y}$ is the observation, it is made of combinations of the components of the hidden state vector $\mathbf{Z}_i$, which is added to the noise vector ε_i.

Now we describe the filtering equations. At each time step $i = 1, \ldots, n$, we start with the *a priori* variables $\overline{Z}_{i|i-1}$ and $P_{i|i-1}$, featuring the estimates of the mean and the variance of Z_i given the observations at time step i and prior to it. For the initialization at step 1, one needs initial values and we choose $\overline{Z}_{1|0} = 0$ and $P_{1|0} = Q_{\eta_1}$, which is defined by (10). Yet the choice of the initial conditions never precludes the convergence of the parameters.

Then we compute the *Kalman gain* K_i, given by

$$K_i = P_{i|i-1} F_i' G_i^{-1}, \quad G_i = F_i P_{i|i-1} F_i' + v^2 I_{L_i+1}$$

where for any matrix M, M' denotes its transpose. The estimate of $\Delta_i^n \mathbf{Y}$ conditionally to the past is given by

$$\overline{\Delta_i^n \mathbf{Y}} = M_i + F_i \overline{Z}_{i|i-1} ,$$

then the *Kalman residual* r_i is given by $r_i = \Delta_i^n \mathbf{Y} - \overline{\Delta_i^n \mathbf{Y}}$: this stands for the difference between the actual observed value and the expectation within the filter. Finally, one can compute the *a posteriori* variables

$$\begin{aligned} \overline{Z}_i &= \overline{Z}_{i|i-1} + K_i r_i , \\ P_i &= \big(I_{2N} - K_i F_i \big) P_{i|i-1} . \end{aligned}$$

And one can prepare the a priori variables of the next step $i + 1$ by computing

$$\begin{aligned} \overline{Z}_{i+1|i} &= A_{i+1} \overline{Z}_i , \\ P_{i+1|i} &= A_{i+1} P_i A_{i+1}' + Q_{\eta_{i+1}} . \end{aligned}$$

It turns out that the Kalman residuals at two different steps are independent from each other, and that $r_i \sim \mathcal{N}(0, G_i)$. After iterating over all time steps, one can compute the likelihood of the sample $(r_1, \ldots, r_n)$ of residuals as

$$\prod_{i=1}^{n} \frac{1}{(2\pi)^{(L_i+1)/2} \det(G_i)^{1/2}} \exp\left(-\frac{1}{2} r_i' G_i^{-1} r_i \right) .$$

Therefore the negative log-likelihood is given by $\mathcal{L}_n + c$, where

$$\mathcal{L}_n = \frac{1}{2} \sum_{i=1}^{n} \log(\det(G_i)) + r_i' G_i^{-1} r_i$$

and c is a real number which does not depend on the parameters.

As we are able to compute the negative log-likelihood for any set of parameters by going through the n filtering steps we described above, we use an optimization routine to minimize $\mathcal{L}_n$. However, leading the optimization over the pristine parameters v, $\alpha_1 < \cdots < \alpha_N$, $\sigma_1 > 0, \ldots, \sigma_N > 0$ and $-1 < \rho_{k,k'} < 1$ for $1 \leq k, k' \leq N$ that we have introduced in Sect. 2 is difficult as one has to handle constraints about the α coefficients being ordered, the σ coefficients being positive and the correlation ones being bounded above and below. Instead, we let

$$a_1 = \log(\alpha_1) \text{ and } a_k = \log(\alpha_k - \alpha_{k-1}) \text{ for } 2 \leq k \leq N \ ,$$

and also we introduce the lower triangular matrix $S = (s_{k,k'})_{1 \leq k,k' \leq N}$ such that SS' is the Cholesky decomposition of the positive-definite matrix

$$\begin{pmatrix} \sigma_1^2 & \rho_{1,2}\sigma_1\sigma_2 & \ldots & \rho_{1,N}\sigma_1\sigma_N \\ \rho_{1,2}\sigma_1\sigma_2 & \sigma_2^2 & \ldots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{1,N}\sigma_1\sigma_N & \ldots & \ldots & \sigma_N^2 \end{pmatrix} .$$

We thus run an unconstrained optimization algorithm over the real numbers v, $a_1, \ldots, a_N$ and $s_{k,k'}$, $1 \leq k' \leq k \leq N$. While this problem still has dimension $\frac{(N+1)(N+2)}{2}$, it is easier to solve numerically as it embeds no constraint at all. We use the method of Nelder and Mead [29], which is a standard simplex method without derivatives.

3.4 Criteria: AIC and BIC

The number of degrees of freedom (to account for in the AIC and BIC) is a function of the number N of factors in the model. For a definition and a discussion of AIC and BIC, we refer the reader to [6]. Given a fixed N, the parameters are as follows: N degrees of freedom corresponding to the parameters a_k, 1 degree of freedom corresponding to parameter v, and $\frac{N(N+1)}{2}$ degrees of freedom corresponding to the coefficients in the lower triangular matrix S. The total number of degrees of freedom is then $(N+1)(N+2)/2$. Therefore, in the case of the factorial models described above, the AIC and BIC are given by:

$$AIC = (N+1)(N+2) + 2\widehat{\mathcal{L}}_n \ ,$$

$$BIC = \frac{1}{2}(N+1)(N+2)\log(n) + 2\widehat{\mathcal{L}}_n \ ,$$

with $\widehat{\mathcal{L}}_n$ denoting the minimized negative log-likelihood of the Kalman residuals that one obtains when applying Kalman filtering to the state-space equation (9).

4 Estimation Results

4.1 Data and Preprocessing Description

In this section, we describe the datasets that we used and we explain how we preprocessed them before running our estimation procedures. We also detail the process of seasonal adjustement of spot prices.

4.1.1 Description of the Data

We have used data of prices in several European power markets, namely Belgium, France, and Germany. Those prices are available on the websites https://eex.com (for the futures prices) and https://epexspot.com (for the spot prices). We collected the closing futures prices every business day, from 2018–01–01 to 2022–11–30, for various contracts, delivering 1 MWh of electricity over standardised periods. Those periods can be:

- the nearest (or 2nd nearest, or 3rd nearest…) week (from Monday to Sunday) that has not begun yet. The underlying contracts are named 1 week-ahead (hereafter 1WAH), 2WAH, 3WAH, … ;
- the nearest months that have not begun yet, corresponding to month-ahead contracts (hereafter MAH);
- the nearest quarters (January–March, April–June, July–September, October–December) that have not begun yet, corresponding to quarter-ahead (QAH) contracts;
- the nearest calendar years that have not begun yet, featuring year-ahead (YAH) contracts.

For each of the previous time spans, a given number of contracts are traded. For every market we collected the following futures contracts: 1 to 4WAH, 1 to 6MAH, 1 to 4QAH and 1 to 2YAH. This leads to 16 futures contracts considered for the estimation. Concerning spot prices, 24 prices are issued every day, related to deliveries of 1 MWh over each of the 24 hours of the day after. We computed the average of those 24 prices each day, featuring the price of the delivery of 1 MWh over the following day. In total we thus considered 17 daily prices for the estimation.

We note that some data are missing, but the maximisation of the likelihood with the Kalman filter as described in Sect. 3.3 can easily deal with a set of missing data. Indeed, in both cases one only has to consider a varying vector size L_i of available prices at each date t_i.

4.1.2 Preprocessing of Data

As noticed in Sect. 3.1, we preprocess the data in order to remove all overlaps in the futures' delivery periods. To do so, we use the structure of futures contracts in power markets and the no arbitrage principle: consider 2 futures contracts $F_t(T_1, \theta_1)$ and $F_t(T_2, \theta_2)$ oberved at date t such that $[T_1, T_1 + \theta_1] \cap [T_2, T_2 + \theta_2] \neq \emptyset$ and $\theta_1 < \theta_2$; we consider that futures contracts with shorter delivery periods give more precise information. In the case $T_1 \leq T_2$, we therefore replace $F_t(T_2, \theta_2)$ by the futures contract $F_t(T_1 + \theta_1, T_2 + \theta_2 - T_1 - \theta_1)$ whose delivery period is disjoint from $[T_1, T_1 + \theta_1]$ and whose value is obtained by no arbitrage:

$$\frac{T_1 + \theta_1 - T_2}{\theta_2} F_t(T_1, \theta_1) + \frac{T_2 + \theta_2 - T_1 - \theta_1}{\theta_2} F_t(T_1 + \theta_1, T_2 + \theta_2 - T_1 - \theta_1)$$
$$= F_t(T_2, \theta_2) \ .$$

In the case $T_1 > T_2$ we replace $F_t(T_2, \theta_2)$ by two futures contracts $F_t(T_2, T_1 - T_2)$ and $F_t(T_1+\theta_1, T_2+\theta_2-T_1-\theta_1)$ of same value deduced from the same no arbitrage principle.

In particular, this preprocessing allows also to face some observed complete redundancy, as it happens that

- three monthly contracts exactly cover a quarter contract;
- four quarterly contracts exactly cover a calendar contract.

The rule previously described removes the contract with longest delivery period (respectively, in the two previous cases, the quarter and the calendar contracts) on each of those days.

Also, we computed the returns at dates of changing products, caring for the specific change. For example, at a date of a month change, e.g. 2018–02–01, we compute the returns between the $(n + 1)$MAH and the nMAH, both corresponding to the same futures contract, namely March 2018 ($n = 1$) to July 2018 ($n = 6$). By doing so, we optimise the quantity of information available in the data, but we do not have the same number L_i of returns each day. As explained earlier, this is acknowledged and easily dealt with in the computation of the joint distribution of the price changes.

4.1.3 Seasonal Adjustment of Spot Prices

In order to use the estimation procedure described in Sect. 3.1, we have to remove the seasonality from spot prices. To do so, we assume that the daily spot price S_t at time t is given by $S_t = \widetilde{S}_t F_{t_0}(t)$, where $\widetilde{S}_t$ is the residual that is modelled in Sect. 3.1 and $F_{t_0}(t)$ is the seasonality term, which we represent with dummy variables as

$$F_{t_0}(t) = y_{year(t)} m_{month(t)} d_{weekday(t)} \ ,$$

where $year(t)$ refers to the year to which t belongs, $month(t) \in \{1, \ldots, 12\}$ is the number of its month, $weekday(t) \in \{1, \ldots, 7\}$ is the number of its day. Furthermore, we let

$$m_{12} = 12 - \sum_{j=1}^{11} m_j \text{ and } d_7 = 7 - \sum_{j=1}^{6} d_j \ ,$$

so that estimation bears only on the first 11 monthly dummies and on the first 6 daily dummies. All the coefficients are estimated with a Least Square procedure. The spot returns are then computed on the residual $\widetilde{S}_t$.

4.2 Results

For each of the three markets (Belgium, France and Germany), we split the global set of data into three subsets: 2018–2019 (before the Covid crisis), 2020 (Covid period) and 2020–12–01 to 2022–11–30 (hereafter referred to as "2021–2022", crisis period). For each of these nine datasets, we run the likelihood maximisation on power prices, as described in Sect. 3.1, starting with $N = 2$ factors. Then we compute the AIC and BIC, and we keep increasing N until the AIC and the BIC start increasing, which means one has reached the balance between the number of parameters and the quality of representation of the prices. The estimation results are given in Appendix. Figures 1, 2 and 3 show the computed AIC and BIC as a function of the number of factors. Table 1 shows the optimal number N of factors and Tables 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28 show the estimated parameters (mean-reverting α_i's, volatility σ_i's and correlations ρ_{ij}'s.

As a global remark on all three markets, we can observe that the BIC allows us to discriminate and find the optimal number of factors more easily than the AIC, which starts increasing quite later than the BIC due to the lower penalty it puts on the number of parameters in the model. Hereafter we will discuss the optimal number of factors according to the BIC, rather than the AIC, in order to stress the penalty linked to the number of parameters and enforce parsimony. In Table 1, in the second row, we show the optimal number of factors according to the BIC. This optimal number is very similar amongst all countries and historical periods, between 8 and 9. In the third row of Table 1 we show the greatest number N of factors where all the correlations are not close to 1 or -1 ($|\rho_{ij}| < 0.98$, for all $i < j, i = 1, \ldots, N-1$). These N (between 2 and 4) are significantly smaller than the BIC-optimal ones, which suggests it is possible to derive some parsimonious, low dimensional models that account quite well for the behaviour of the prices. The additional factors only allow one to adjust the volatility function in order to compensate the difficulty of the factorial model (with only exponential volatility functions) to represent the observed volatility which may incorporate non-monotonic behaviours.

Table 1 Optimal number N of factors according to the AIC (first row) or the BIC (second row) and greatest N for which all the correlations between two factors are smaller than 0.98

	Belgium			France			Germany		
	18–19	20	21–22	18–19	20	21–22	18–19	20	21–22
AIC-optimal N	9	10	9	10	9	9	10	9	10
BIC-optimal N	8	9	8	9	8	8	9	8	8
Greatest N, max $\lvert\rho_{kk'}\rvert < 0.98$	3	2	3	3	3	4	2	3	3

Within a given market and a given time period, the values taken by a given estimated parameter are rather stable over the number of factors; see for instance Table 3 and choose a given column i: the values of the estimator of α_i remain quite the same as soon as the number of factors in the estimation is greater than i. One can make the same observation with the estimates of the σ_i.

It also turns out that for a fixed time period, the values of the estimated parameters have quite the same order of magnitude from one country to another, especially for α and at the BIC-optimal number of factors.

Also, we can observe similar characteristics of the first factor (i.e. the factor with the smallest value of α), mainly driving the long term volatility), for all countries. In particular, for N close to the BIC-optimal number of factors, one can observe quite stable and similar estimated values of α_1 (between 0.58 in Germany and 1.39 in France) and a slight increase for the period 2021–2022. The estimated value of σ_1 is, for all countries, around 30 and 40% for periods 2018–2019 and 2020, respectively, whereas it explodes for period 2021–2022, highlighting the impact of the European crisis in the energy markets and the fact that the markets are interconnected and share some fundamentals.

Let us discuss a bit the correlation matrices. For all countries and periods, the correlation matrices seem stable along the number of factors. Also, they feature a first factor which is generally weakly correlated to the other factors. When one, for instance, looks at the first column in the correlation matrices in Tables 8, 9, 17, 18, 26 and 27, it appears that very few of the noninitial elements have an absolute value higher than 0.20 for periods 2019–2019 and 2020. Interestingly, we are meeting a different situation in 2021–2022: the first factor is highly anti-correlated to the second factor, see $N = 8$ in Belgium, in France or in Germany. We emphasize another very noticeable pattern, which is that most of the time (at least when $N > 5$) factors with different parities would have a negative correlation, and have a positive one if they share the same parity. As an example, in Table 28, the cases $N = 5$ and $N = 8$ stand for perfect examples of this pattern, which the case $N = 7$ is also quite close to doing so.

We may compare our results to the ones available in the literature. A two-factor model has been calibrated to German option prices on futures contracts in [24] (from the year 2005). In [13] the calibration of the same two-factor model is done on marginal volatilities of futures contracts and on spot prices in the French and UK markets, using data from 2013 to 2015. In both articles the authors fixed α_1 and the correlation between the two factors to 0. The results obtained in the present paper