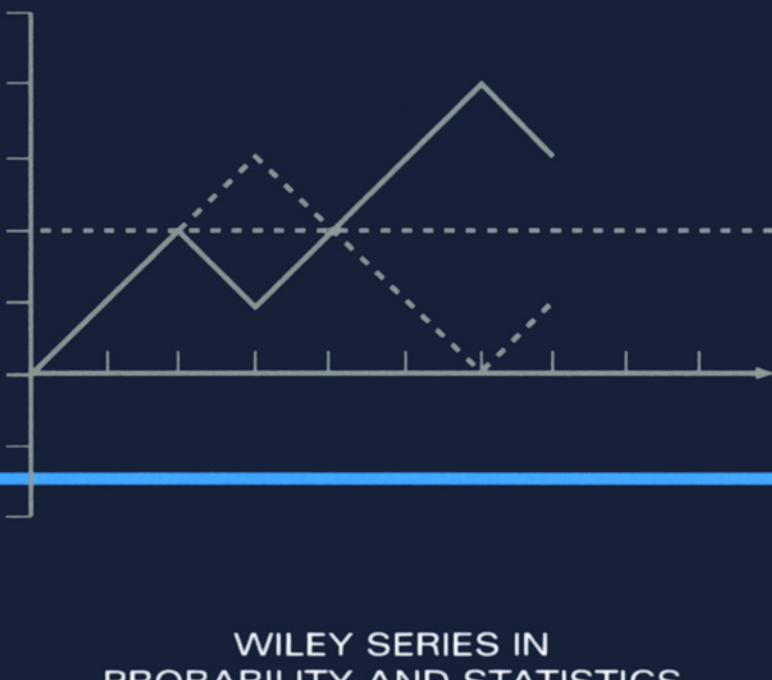
Probability

A Survey of the Mathematical Theory

Second Edition

John W. Lamperti



WILEY SERIES IN
PROBABILITY AND STATISTICS

This page intentionally left blank

Probability

WILEY SERIES IN PROBABILITY AND STATISTICS

Established by WALTER A. SHEWHART and SAMUEL S. WILKS

A complete list of the titles in this series appears at the end of this volume

Probability

A Survey of the Mathematical Theory

Second Edition

JOHN W. LAMPERTI

Wiley Series in Probability and Statistics

A Wiley-Interscience Publication
JOHN WILEY & SONS, INC.
New York • Chichester • Brisbane • Toronto • Singapore

This text is printed on acid-free paper.

Library of Congress Cataloging in Publication Data:

Lamperti, J. (John)
Probability : a survey of the mathematical theory / John W. Lamperti. — 2nd ed.
p. cm. — (Wiley series in probability and statistics)
"A Wiley Interscience publication."
Includes bibliographical references (p. –) and index.
ISBN 13: 978-0-471-15407-5
1. Probabilities. I. Title. II. Series: Wiley series in probability and statistics. Probability and statistics.
QA273.L26 1996
519.2—dc20 96-9559

10 9 8 7 6 5 4 3 2 1

Contents

Preface

The first edition of this book resulted from a one-semester graduate course I taught at Dartmouth College. The book, like the course, offered an overview of "classical" probability theory (up to around 1950), with results about sums of independent random variables occupying center stage. The book made some friends in its day, and I still hear occasionally from readers who found it helpful, or from teachers who wish to use it in a course.

That first edition has been out of print for several years, and I hope this new version will prove useful to teachers and students of our subject. It is fundamentally the same book, but entirely rewritten and, I hope, substantially improved. It has not been watered down. I have attempted to improve the exposition and, in places, to help the reader with better motivation and a few more examples. As in the original edition, I have tried to be honest but brief in the necessary use of measure theory, keeping the focus on the probabilistic ideas. Most importantly, perhaps, the book is still informal and short. A reader who finishes it will not know all there is to know about any part of the field; but, in compensation, that reader can begin section one with the expectation of finishing the book in a reasonable period of time, and then go on to current problems of his or her choosing.

Differences from the first edition include new sections on conditional probability in Chapter 1 and on the Markov concept in Chapter 4, plus a brief appendix summarizing necessary ideas and tools from measure theory. In Chapter 3 the reader will now find an account of the elegant parallel between limiting distributions for sums and for maxima of independent random variables. On the other hand, I have removed from Chapter 4 the sections about Markov transition functions which can be found in my book (or another) on stochastic processes and which belong more naturally there. (See the bibliography.) There are also many smaller differences in proofs, exposition, and problems.

As before, this book is not intended to provide anyone's first acquaintance with probability. (William Feller's classic *Introduction* is still my favorite for getting started in the field, but today there are also other good choices.) The other prerequisite, in addition to elementary probability, is knowledge of analysis at roughly beginning graduate level; Halsey Royden's *Real Analysis* contains all that's needed in this area and more, as do many other books. Of course, some willingness to think things through for oneself is also essential for best results!

The need to think for oneself pertains not only to the study of mathematics proper, but also to the ways in which mathematics and science are used. I believe that each of us has a measure of responsibility for the results of our work, a responsibility which cannot be simply passed along to employers or to governments. The need to consider the consequences is most evident for applied science and engineering—and, of course, probability theory has applications, for good or ill, in a great many areas of human activity. But the scientist's responsibility applies to the practice of "pure" mathematics as well. Through both teaching and research we are part of a collective enterprise with enormous social consequences, and the alleged "uselessness" of a particular theorem or concept does not shelter us from a share of personal involvement in how mathematics and science are put to work.

While working on this book I learned that the 1995 Nobel Prize for Peace had been awarded to the physicist Joseph Rotblat and the Pugwash Conferences on Science and World Affairs which he helped to found. In accepting the prize, Rotblat commented:

> I want to speak as a scientist, but also as a human being. From my earliest days I had a passion for science. But science, the exercise of the supreme power of the human intellect, was always linked in my mind with benefit to people. I saw science as being in harmony with humanity. I did not imagine that the second half of my life would be spent on efforts to avert a mortal danger to humanity created by science. [The danger is nuclear war.]
>
> At a time when science plays such a powerful role in the life of society, when the destiny of the whole of mankind may hinge on the results of scientific research, it is incumbent on all scientists to be fully conscious of that role, and conduct themselves accordingly. I appeal to my fellow scientists to remember their

responsibility to humanity. [Quoted from *The Bulletin of the Atomic Scientists* 52 (March/April 1996), pp. 26–28.]

I urge the readers of this book to remember Joseph Rotblat's appeal, and I hope we can work together to put it into practice.

JOHN W. LAMPERTI
Hanover, N.H.
April, 1996

A WORD ON NOTATION

Equations, theorems, and problems are numbered within each section simply as 1, 2, 3, etc. A reference to "Theorem n," for example, always means a theorem in the current section of the book, whereas "Theorem 15.3" refers to Theorem 3 in Section 15. References to problems and equations are treated the same way, so that "(8.4)" indicates the fourth numbered equation of Section 8.

Probability

This page intentionally left blank

CHAPTER ONE

Foundations

1. PROBABILITY SPACES

Let Ω be any nonempty set and suppose that $\mathcal{B}$ is a *Borel field* or, equivalently, a *σ-field* (pronounced "sigma field") of subsets of Ω. This means that $\mathcal{B}$ is a collection of subsets that contains the empty set $\varnothing$ and is closed under the formation of complements and of finite or countable unions of its members. Let P be a nonnegative function defined on $\mathcal{B}$ such that $P(\Omega) = 1$ and which is *countably additive* in the sense that

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n) \tag{1}$$

provided $A_n \in \mathcal{B}$ and $A_n \cap A_m = \varnothing$ for each $n \neq m$.

Definition 1. *Under these conditions P is a* probability measure *and the triple $(\Omega, \mathcal{B}, P)$ is a* probability space. *Sets belonging to $\mathcal{B}$ are called* events.

This definition generalizes the discrete probability spaces that suffice for many applications and are commonly used to introduce the subject.

Problem 1. *Suppose $\Omega = \{\omega_n\}$ is a finite or countably infinite set, and let $\mathcal{B}$ denote the collection of all its subsets. Assume that a nonnegative*