

CALC
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SIMPLIFIED

OSCAR E. FERNANDEZ

CALCULUS

Simplified

Other Books by Oscar Fernandez

Everyday Calculus: Discovering the Hidden Math All Around Us

*The Calculus of Happiness: How a Mathematical Approach to Life Adds Up to
Health, Wealth, and Love*

CALCULUS *Simplified*

Oscar E. Fernandez

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To Emilia and Alicia

*Many years from now when you confront calculus,
Come back to this book and give it a read.
Come back to me, too, and give me a hug.
As much as I love math, I love you both much, much more.*

Contents

<i>Preface</i>	<i>xi</i>
<i>To the Student</i>	<i>xvii</i>
<i>To the Instructor</i>	<i>xix</i>
<i>Before You Begin . . .</i>	<i>xxi</i>

1	The Fast Track Introduction to Calculus	1
1.1	What Is Calculus?	1
	Calculus as a Way of Thinking	1
	What Does “Infinitesimal Change” Mean?	2
1.2	Limits: The Foundation of Calculus	3
1.3	The Three Difficult Problems That Led to the Invention of Calculus	5
2	Limits: How to Approach Indefinitely (and Thus Never Arrive)	8
2.1	One-Sided Limits: A Graphical Approach	8
2.2	Existence of One-Sided Limits	11
2.3	Two-Sided Limits	13
2.4	Continuity at a Point	15
2.5	Continuity on an Interval	17
2.6	The Limit Laws	21
2.7	Calculating Limits—Algebraic Techniques	25
2.8	Limits Approaching Infinity	30
2.9	Limits Yielding Infinity	33
2.10	Parting Thoughts	37
	<i>Chapter 2 Exercises</i>	<i>37</i>
3	Derivatives: Change, Quantified	43
3.1	Solving the Instantaneous Speed Problem	43
3.2	Solving the Tangent Line Problem—The Derivative at a Point	47
3.3	The Instantaneous Rate of Change Interpretation of the Derivative	50
3.4	Differentiability: When Derivatives Do (and Don’t) Exist	51

3.5	The Derivative, a Graphical Approach	53
3.6	The Derivative, an Algebraic Approach	55
	Leibniz Notation	59
3.7	Differentiation Shortcuts: The Basic Rules	60
3.8	Differentiation Shortcuts: The Power Rule	61
3.9	Differentiation Shortcuts: The Product Rule	64
3.10	Differentiation Shortcuts: The Chain Rule	65
3.11	Differentiation Shortcuts: The Quotient Rule	68
3.12	(Optional) Derivatives of Transcendental Functions	69
3.13	Higher-Order Derivatives	74
3.14	Parting Thoughts	75
	<i>Chapter 3 Exercises</i>	76
4	Applications of Differentiation	82
4.1	Related Rates	82
4.2	Linearization	89
4.3	The Increasing/Decreasing Test	93
4.4	Optimization Theory: Local Extrema	98
4.5	Optimization Theory: Absolute Extrema	101
4.6	Applications of Optimization	106
4.7	What the Second Derivative Tells Us About the Function	112
4.8	Parting Thoughts	117
	<i>Chapter 4 Exercises</i>	118
5	Integration: Adding Up Change	125
5.1	Distance as Area	125
5.2	Leibniz's Notation for the Integral	128
5.3	The Fundamental Theorem of Calculus	130
5.4	Antiderivatives and the Evaluation Theorem	133
5.5	Indefinite Integrals	135
5.6	Properties of Integrals	138
5.7	Net Signed Area	139
5.8	(Optional) Integrating Transcendental Functions	141
5.9	The Substitution Rule	143
5.10	Applications of Integration	148
5.11	Parting Thoughts	152
	<i>Chapter 5 Exercises</i>	153
	<i>Epilogue</i>	159
	<i>Acknowledgments</i>	161
	<i>Appendix A: Review of Algebra and Geometry</i>	163

<i>Appendix B: Review of Functions</i>	177
<i>Appendix C: Additional Applied Examples</i>	215
<i>Answers to Appendix and Chapter Exercises</i>	227
<i>Bibliography</i>	239
<i>Index of Applications</i>	241
<i>Index of Subjects</i>	243

Preface

Hi. Welcome to *Calculus Simplified*. My name is Oscar Fernandez, Associate Professor of Mathematics and Faculty Director of the Pforzheimer Learning and Teaching Center at Wellesley College, and I will be your instructor.

Who Is This Book Intended For?

Here are three questions that will help you determine if this book is for *you*.

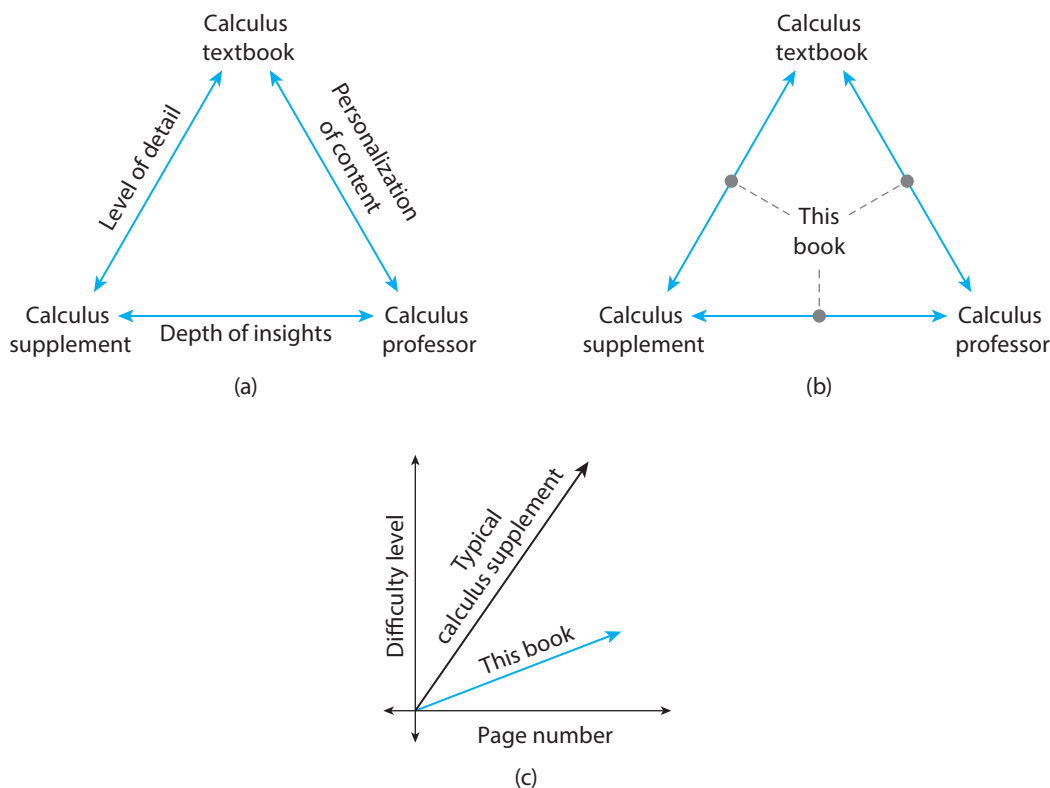
- **Do you have a background in algebra, geometry, and some exposure to functions (exposure to transcendental functions—exponentials, logs, trigonometry—is not required)?** If so, this book is for you.
- **Are you currently enrolled in a calculus course (or soon will be)?** If so, this book is for you.
- **Did you learn calculus long ago and are now seeking a quick refresher on the subject?** If so, this book is for you.

If you answered “no” to all of those questions, this book *might* not be for you. I encourage you to skim it first to see if it may still be an appropriate resource for you. If you answered “yes” to any of those questions, great! Read on.

Reason 1 You Should Use This Book: Its Goldilocks Approach to Calculus

Cognitive scientists have accumulated evidence over the past few decades supporting what is today called the “Goldilocks Effect”: we learn best when the content being taught contains just the right amount of challenge and complexity—not too much, and not too little.

Consider now the challenge of learning calculus. The typical calculus student turns to three particular resources for help: a calculus textbook, a calculus professor, and a calculus supplement. Each of these resources, however, has its strengths and weaknesses. I’ve highlighted three particular dimensions along which to understand those strengths and weaknesses in (a) of Figure 1: level of detail, personalization of content, and depth of insights.

**Figure 1**

Compare the aforementioned three calculus resources along those three dimensions and here are some of the things you'd notice:

- **Regarding the level of detail.** Most calculus supplements (e.g., *Calculus for Dummies*) are devoid of formal statements of theorems. That means it isn't always clear when one can apply the formulas and techniques discussed (this is clearly articulated by the hypotheses of a theorem). Most calculus textbooks, on the other hand, have the opposite problem—they are replete with formal statements of theorems (and their proofs). The result is a calculus learning experience that feels too formal, where proof and rigor often overcomplicate explanations and obscure the intuition behind the concepts. Conclusion: too little detail may give you a false sense of confidence in your calculus knowledge; too much detail may turn you off from calculus altogether.
- **Regarding the depth of insights.** Most calculus supplements offer only superficial mathematical insights, focusing instead more on teaching computational skills, procedures, and techniques. (Example: "Do *this* when you see *that*.") In fairness, most calculus supplements are *supplements*; the idea is

to use them alongside a calculus textbook and/or calculus professor, which furnish those deeper mathematical insights. Often these resources go too far, however, as evidenced by one of the most common nuggets of student feedback in a calculus course: “less theory, more examples, please.” Conclusion: too little depth of insight will make learning calculus feel like rote learning; too much depth of insight makes calculus feel too theoretical and impractical.

- **Regarding the personalization of content.** Most calculus textbooks are thousand-page tomes containing way more content than any calculus professor can cover in a Calculus 1 course. The average calculus textbook, therefore, is not at all personalized to your interests. We calculus professors do our best to distill the hundreds of pages of content provided by the calculus textbook into roughly 30 hour-long lessons, ideally taking into consideration the particular interests of the students in the course. This is an improvement over the calculus textbook, for sure. But in a class of *many* students it is still hard to personalize the content to *each* student. Conclusion: too little personalization of content is a wasted opportunity to engage you in learning calculus; however, the amount the average calculus instructor provides, while an improvement over that of a calculus textbook’s, is still not personalized enough.

The conclusion: *none of the resources just discussed are “just right” for learning calculus.* That’s where this book comes in.

This book takes a “Goldilocks approach” to learning calculus.

As Figure 1(b) is meant to illustrate:

- **This book balances intuition with theory to provide you with just the right level of detail.** Chapter 1 teaches you the core ideas of calculus. What you will learn there will anchor *all* of what you will learn in the rest of the book. That’s because the Chapter focuses on developing the intuition behind the main concepts, mindset, and overarching framework of calculus. Subsequent chapters discuss the math of calculus, with just the right balance of formal statements of definitions and theorems, so that you learn the terminology of calculus and understand the *full* story, complete with when and why it works.
- **This book allows you to personalize your calculus adventure.** First of all:

No prior knowledge of exponential, logarithmic, or trigonometric functions is needed to learn calculus from this book.

Don’t know what $\sin x$ is (or don’t yet fully understand it)? Same issue with e^x or $\ln x$? No problem; the calculus of such functions is left until the end of

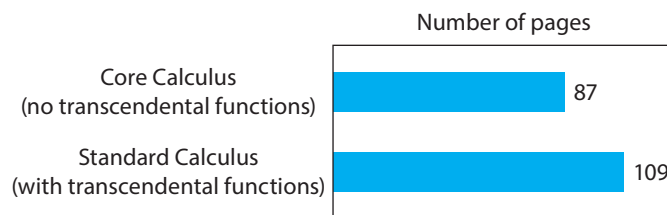


Figure 2

every section (discussed under the heading Transcendental Tales). Include it in your reading if you'd like; skip it if you'd rather not. The choice is yours.

Additionally, sections *do* include discussions of theory, but focus on intuition rather than proof. More technical discussions and proofier content is relegated to the chapter appendixes (which you can download from the book's website). Include it in your reading if you'd like; skip it if you'd rather not. The choice is yours.

Finally, the same approach is used with the more in-depth real-world applications. These are included in Appendix C. Include it in your reading if you'd like; skip it if you'd rather not. The choice is yours.

The net effect of all this: a gentler learning experience, as illustrated in Figure 1(c), and a focus on intuition (hence the *Calculus Simplified* title).

- **This book provides just the right amount of depth to the mathematical insights unearthed.** You will learn both the “how” *and* the “why” of calculus. You will understand why its core concepts are important. You will be exposed to the various other places (e.g., real-world contexts) where calculus concepts show up. And if you want to, you will also understand the historical context that motivated the invention of calculus (some of that content is optional and contained in the chapter appendixes).

Reason 2 You Should Use This Book: The Bonus Features

The Goldilocks approach used in this book is complemented by the following additional features designed to supercharge your learning of calculus.

- **A focus on conciseness.** Excluding the exercises and appendixes:

This book teaches you calculus in at most 110 pages.

Exclude transcendental functions (exponentials, logarithms, and trigonometric functions) and you're down to 87 pages (Figure 2). Both of these

numbers are considerably less than the roughly 200–300 pages the typical calculus textbook takes to teach the same content (again excluding the exercises and appendixes).

- **Nearly 200 solved examples.** I’ve included 196 distinct solved examples in this book. I also included more than just the calculations for many of them—I wrote out my thought process too. This will help you learn to think like a mathematician thinks about calculus.
- **Answers to *all* non-proof exercises.** There are 337 exercises in this book. I have included answers to *all* of the exercises that are not proofs or derivations; answers start on page 227.
- **Use of color and boxes to separate content.** In-chapter definitions, theorems, and important takeaways appear in blue boxes to help you easily spot them. End-of-chapter exercises are also color-coded: exercises colored blue are applied exercises. And of course, colors are used to help explain various concepts.
- **Interactive online content linked to book content.** In mathematics, an interactive graph—or a video lesson—is sometimes worth a million words. I’ve capitalized on the learning power of these digital resources by creating several ones based on the content in this book. You’ll find references to them throughout the book; a link to the graphs and other digital resources can be accessed on the book’s website:

<https://press.princeton.edu/titles/13351.html>

- **Inclusion of references.** Sources for external content used—and for additional useful resources—are provided in the Bibliography. References appear in-text as brackets and use a number (e.g., [3]) to identify the entry in the bibliography being referenced.
- **A direct line to the author.** I wrote this book to help you learn calculus. I stand by this goal. So, *feel free to email me with any questions, comments, or suggestions*. Seriously. Here is my email:

math@surroundedbymath.com

I also encourage you to submit feedback on the book via the link below.

<https://goo.gl/forms/yOIFolqTEEdzkVhr2>

Your submission will be anonymous. Your feedback will help me improve the book and will be incorporated into future editions.

Parting Thoughts

Calculus Simplified is a resource for anyone interested in learning (or relearning) Calculus 1. First and foremost, it is an attempt to re-structure the standard presentation of content in a calculus textbook to strike that “just right” balance on the level of detail, depth of insights, and personalization of content illustrated in Figures 1(a) and (b).

Second, *Calculus Simplified* was designed to streamline your learning of calculus. But do not confuse “streamline” with “water down.” This book is not a collection of calculus formulas or merely a quick review of calculus concepts (which presumes you already know calculus). It is not an idiot’s or dummies guide to calculus (you are neither). *Calculus Simplified* is a college-level Calculus 1 course—based on the notes I created for teaching calculus—streamlined to eliminate excess content that gets in the way of learning calculus, written in more informal prose, including relevant real-world examples, and structured to afford you multiple routes to learning calculus.

Finally, though the vast majority of topics covered in a calculus textbook are also covered in *Calculus Simplified*, this book is not intended to be a comprehensive treatment of Calculus 1. In the present conception of what a calculus textbook consists of, this book is also not intended to be a textbook (though it can certainly be used as such in some settings). At the same time, *Calculus Simplified* is much more than the run-of-the-mill calculus supplement. Explained within the theme of this Preface, I think of *Calculus Simplified* as occupying the “Goldilocks zone” between a calculus textbook and a calculus supplement.

I am excited to begin working with you as you set off on your calculus adventure. Once you are done learning the calculus in this book, I encourage you to read the Epilogue; it contains some useful advice and encouragement for navigating mathematics beyond calculus. See you in Chapter 1!

Oscar E. Fernandez
Brookline, MA

To the Student

Welcome to *Calculus Simplified*! Before you embark on your adventure through this book, I thought I would give you a few practical tips intended to help you conquer calculus.

What You Can Expect to Learn from This Book

This subheading's title might seem silly, given that this is a book about calculus. But research shows that students learn best when they know *beforehand* what they are about to learn and what they should be able to do with that knowledge at the end of the lesson. Rather than insert these learning goals and objectives at the start of each lesson, I have provided them in a supplemental document titled *Calculus Simplified: Learning Goals and Objectives*, available on the book's website. (That document also maps this book's learning goals and objectives to the curriculum framework used in AP Calculus, in case you're enrolled in such a course.) I highly recommend you keep that document handy as you read through each section of the book.

How to Read a Math Book (Including This One)

Though I have done my best to infuse my writing with the elements of a novel—characters, a plot, etc.—this book is not a novel. One thing this implies is that you need to read this book differently than you would a novel. For example, simply *reading* this book won't help you understand calculus. Rather, I recommend you *work through* this book—work out the examples, work out the solutions to the exercises, work through the supplemental content. By *doing* mathematics you will be helping yourself *learn* mathematics. Moreover, jot down questions and comments as you read and work through this book. This will ensure you are learning *actively* rather than passively.

Lastly, let me mention the special role that theorems play in mathematics, and how to ensure you're getting the most out of them. Loosely speaking, a theorem is a statement that has been proven true. A typical theorem has the following structure: preamble, hypotheses, conclusion. Example:

Theorem (Pythagorean Theorem): Consider a right triangle in the plane. Let c denote the length of the hypotenuse of the triangle and a and b the lengths of the other two sides. Then $c^2 = a^2 + b^2$.

In this theorem the first sentence is the preamble; its role is to provide context for what the theorem says. The second sentence in the theorem contains some assumptions (as happened here, sometimes the preamble also contains assumptions). The last sentence contains the conclusion.

Echoing my earlier advice to work through this book, do the same with theorems—whenever you come across a theorem take a moment to understand what it is saying. Try to draw pictures, explain the theorem in words, and imagine removing some of the hypotheses to see how the conclusion might be affected. Doing all this will help you appreciate what the real use of the theorem is, help you remember it, and help you learn when it can (and cannot) be applied.

How to Become a Better Student

I have one last recommendation for you: *employ the latest research from the science of learning while you study*. Study strategies like retrieval practice and interleaving—both backed by cognitive science research—can supercharge your studying. You can read more about these and other research-backed study strategies in the supplemental document titled *Evidence-Based Study Strategies*, available on the book's website.

Alright, that's all I have for you at the moment. Let's get started with your calculus adventure!

To the Instructor

You might be thinking, “Not another calculus book!” But this one is different. My goal is neither to add another calculus textbook to the volumes of such books nor to provide students with an overly simplistic treatment of the subject. Instead, as I wrote in the preface, *Calculus Simplified* is an attempt to re-structure the standard presentation of content in a calculus textbook to strike that “just right” balance on the level of detail, depth of insights, and personalization of content.

Calculus Simplified is also a sign of the times. In the age of Twitter, it has become increasingly clear that shorter and more succinct treatments of calculus are favored by students. The byte-sized explanations utilized in this book, along with the freedom to add or exclude content related to transcendental functions, cut down the time investment necessary for a student to quickly familiarize herself with a calculus concept. As such, I have used this very text successfully not just as a main text in introductory calculus courses, but also as a quick reference text in more rigorous calculus courses (especially due to its excellent breadth of exercises).

One final note that you may find helpful: the explicit connection to the AP Calculus Mathematical Practices curriculum framework. Even if you are not teaching an AP Calculus course, I highly recommend reading the Fall 2016 revision of the AP Calculus curriculum. In addition to discussing a variety of useful teaching techniques for calculus, the document details six broad learning goals for calculus (these are the Mathematical Practices) along with detailed learning objectives for each concept covered in AP Calculus. On the website that accompanies this book, you will find a supplemental document titled *Calculus Simplified: Learning Goals and Objectives* that maps each of the AP Calculus framework’s learning goals and objectives to the associated sections and exercises in this book. This makes it especially easy to use this book in just-in-time fashion to provide content and exercises for whatever calculus topic you are teaching on a given day.

Before You Begin . . .

Here is some useful information that will help you navigate this book.

Numbering scheme

- Where applicable, equations in the chapters are numbered in (chapter.equation) format. Example: equation (3.17) refers to the 17th numbered equation in Chapter 3.
- Equation numbers appear flushed right in the book, like this: (3.17)
- Figures and tables in the chapters follow the same numbering scheme as equations, except no parentheses surround the “x.y” reference.
- Equations and figures/tables in appendixes A, B, and C are numbered in (appendix.equation) format. Example: equation (B.7) refers to the 7th numbered equation in Appendix B.
- Equations and figures/tables in the appendix to chapter X are numbered as (AX.y) and AX.y, respectively. Example: equation (A1.5) refers to the 5th numbered equation in the appendix to Chapter 1.

Color-coded alerts

- Definitions and theorems appear next to thin blue vertical rectangles, like this:

Theorem 3.2 Theorem text . . .

- Appendix C contains applied examples too long to include in the chapters. These are referenced with light blue rectangles, like this:

Reference Text Short description of applied example.

- Suggested end-of-chapter exercises are referenced like this:

Related Exercises 3, 4, etc.

- Thin blue-colored rectangles in the margin—like the one to the right—alert you to online interactive versions of the graph(s) or content being discussed (accessible via the book’s website, listed in the Preface).

- Content involving transcendental functions (exponential, logarithmic, and trigonometric functions) appears under the subheading **Transcendental Tales**.
- In many sections I suggest tips and summarize takeaways; these appear under the subheading **Tips, Tricks, and Takeaways**.

CALCULUS

Simplified

1 The Fast Track Introduction to Calculus

Chapter Preview. *Calculus is a new way of thinking about mathematics. This chapter provides you with a working understanding of the calculus mindset, core concepts of calculus, and the sorts of problems they help solve. The focus throughout is on the ideas behind calculus (the big picture of calculus); the subsequent chapters discuss the math of calculus. After reading this chapter, you will have an intuitive understanding of calculus that will ground your subsequent studies of the subject. Ready? Let's start the adventure!*

1.1 What Is Calculus?

Here's my two-part answer to that question:

Calculus is a mindset—a dynamics mindset. Contentwise, calculus is the mathematics of infinitesimal change.

Calculus as a Way of Thinking

The mathematics that precedes calculus—often called “pre-calculus,” which includes algebra and geometry—largely focuses on *static* problems: problems devoid of change. By contrast, change is central to calculus—calculus is about *dynamics*. Example:

- What's the perimeter of a square of side length 2 feet? ← *Pre-calculus problem.*
- How fast is the square's perimeter *changing* if its side length is *increasing* at the constant rate of 2 feet per second? ← *Calculus problem.*

This statics versus dynamics distinction between pre-calculus and calculus runs even deeper—change is the *mindset* of calculus. The subject trains you to think of a problem in terms of dynamics (versus statics). Example:

- Find the volume of a sphere of radius r . *Pre-calculus mindset:* Use $\frac{4}{3}\pi r^3$ (Figure 1.1(a)).
- Find the volume of a sphere of radius r . *Calculus mindset:* Slice the sphere into a gazillion disks of tiny thickness and then add up their volumes (Figure 1.1(b)). When the disks' thickness is made “infinitesimally small” this approach reproduces the $\frac{4}{3}\pi r^3$ formula. (We will discuss why in Chapter 5.)

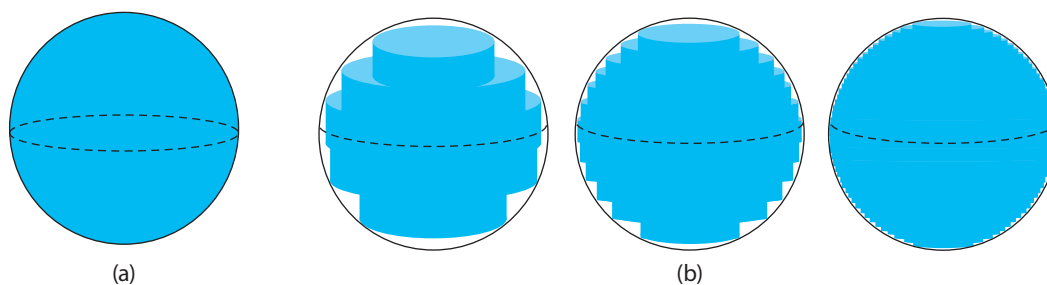


Figure 1.1: Visualizing the volume of a sphere via (a) a pre-calculus mindset and (b) a calculus mindset.

There’s that mysterious word again—infinitesimal—and I’ve just given you a clue of what it might mean. I’ll soon explain. Right now, let me pause to address a thought you might have just had: “Why the slice-and-dice approach? Why not just use the $\frac{4}{3}\pi r^3$ formula?” The answer: had I asked for the volume of some random blob in space instead, that static pre-calculus mindset wouldn’t have cut it (there is no formula for the volume of a blob). The dynamics mindset of calculus, on the other hand, would have at least led us to a reasonable approximation using the same slice-and-dice approach.

That volume example illustrates the power of the dynamics mindset of calculus. It also illustrates a psychological fact: *shedding the static mindset of pre-calculus will take some time*. That was the dominant mindset in your mathematics courses prior to calculus mathematics courses, so you’re accustomed to thinking that way about math. But fear not, young padawan (a *Star Wars* reference), I am here to guide you through the transition into calculus’ dynamics mindset. Let’s continue the adventure by returning to what I’ve been promising: insight into infinitesimals.

What Does “Infinitesimal Change” Mean?

The volume example earlier clued you in to what “infinitesimal” might mean. Here’s a rough definition:

“Infinitesimal change” means: *as close to zero change as you can imagine, but not zero change.*

Let me illustrate this by way of Zeno of Elea (c. 490–430 BC), a Greek philosopher who devised a set of paradoxes arguing that motion is not possible. (Clearly, Zeno did not have a dynamics mindset.) One such paradox—the Dichotomy Paradox—can be stated as follows:

To travel a certain distance you must first traverse half of it.

Figure 1.2 illustrates this. Here Zeno is trying to walk a distance of 2 feet. But because of Zeno’s mindset, with his first step he walks only half the distance: 1 foot