

Advanced Stochastic Models, Risk Assessment, and Portfolio Optimization

*The Ideal Risk, Uncertainty,
and Performance Measures*

SVETLOZAR T. RACHEV
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FRANK J. FABOZZI



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Published by John Wiley & Sons, Inc., Hoboken, New Jersey.

Published simultaneously in Canada.

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ISBN: 978-0-470-05316-4

Printed in the United States of America.

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STR

To my children, Boryana and Vladimir

SVS

*To my parents, Veselin and Eugeniya Kolevi, and my
brother, Pavel Stoyanov*

FJF

*To the memory of my parents,
Josephine and Alfonso Fabozzi*

Contents

Preface	xiii
Acknowledgments	xv
About the Authors	xvii
CHAPTER 1	
Concepts of Probability	1
1.1 Introduction	1
1.2 Basic Concepts	2
1.3 Discrete Probability Distributions	2
1.3.1 Bernoulli Distribution	3
1.3.2 Binomial Distribution	3
1.3.3 Poisson Distribution	4
1.4 Continuous Probability Distributions	5
1.4.1 Probability Distribution Function, Probability Density Function, and Cumulative Distribution Function	5
1.4.2 The Normal Distribution	8
1.4.3 Exponential Distribution	10
1.4.4 Student's t -distribution	11
1.4.5 Extreme Value Distribution	12
1.4.6 Generalized Extreme Value Distribution	12
1.5 Statistical Moments and Quantiles	13
1.5.1 Location	13
1.5.2 Dispersion	13
1.5.3 Asymmetry	13
1.5.4 Concentration in Tails	14
1.5.5 Statistical Moments	14
1.5.6 Quantiles	16
1.5.7 Sample Moments	16
1.6 Joint Probability Distributions	17
1.6.1 Conditional Probability	18
1.6.2 Definition of Joint Probability Distributions	19

1.6.3	Marginal Distributions	19
1.6.4	Dependence of Random Variables	20
1.6.5	Covariance and Correlation	20
1.6.6	Multivariate Normal Distribution	21
1.6.7	Elliptical Distributions	23
1.6.8	Copula Functions	25
1.7	Probabilistic Inequalities	30
1.7.1	Chebyshev's Inequality	30
1.7.2	Fréchet-Hoeffding Inequality	31
1.8	Summary	32
CHAPTER 2		
	Optimization	35
2.1	Introduction	35
2.2	Unconstrained Optimization	36
2.2.1	Minima and Maxima of a Differentiable Function	37
2.2.2	Convex Functions	40
2.2.3	Quasiconvex Functions	46
2.3	Constrained Optimization	48
2.3.1	Lagrange Multipliers	49
2.3.2	Convex Programming	52
2.3.3	Linear Programming	55
2.3.4	Quadratic Programming	57
2.4	Summary	58
CHAPTER 3		
	Probability Metrics	61
3.1	Introduction	61
3.2	Measuring Distances: The Discrete Case	62
3.2.1	Sets of Characteristics	63
3.2.2	Distribution Functions	64
3.2.3	Joint Distribution	68
3.3	Primary, Simple, and Compound Metrics	72
3.3.1	Axiomatic Construction	73
3.3.2	Primary Metrics	74
3.3.3	Simple Metrics	75
3.3.4	Compound Metrics	84
3.3.5	Minimal and Maximal Metrics	86
3.4	Summary	90
3.5	Technical Appendix	90

3.5.1	Remarks on the Axiomatic Construction of Probability Metrics	91
3.5.2	Examples of Probability Distances	94
3.5.3	Minimal and Maximal Distances	99

CHAPTER 4

Ideal Probability Metrics **103**

4.1	Introduction	103
4.2	The Classical Central Limit Theorem	105
4.2.1	The Binomial Approximation to the Normal Distribution	105
4.2.2	The General Case	112
4.2.3	Estimating the Distance from the Limit Distribution	118
4.3	The Generalized Central Limit Theorem	120
4.3.1	Stable Distributions	120
4.3.2	Modeling Financial Assets with Stable Distributions	122
4.4	Construction of Ideal Probability Metrics	124
4.4.1	Definition	125
4.4.2	Examples	126
4.5	Summary	131
4.6	Technical Appendix	131
4.6.1	The CLT Conditions	131
4.6.2	Remarks on Ideal Metrics	133

CHAPTER 5

Choice under Uncertainty **139**

5.1	Introduction	139
5.2	Expected Utility Theory	141
5.2.1	St. Petersburg Paradox	141
5.2.2	The von Neumann–Morgenstern Expected Utility Theory	143
5.2.3	Types of Utility Functions	145
5.3	Stochastic Dominance	147
5.3.1	First-Order Stochastic Dominance	148
5.3.2	Second-Order Stochastic Dominance	149
5.3.3	Rothschild–Stiglitz Stochastic Dominance	150
5.3.4	Third-Order Stochastic Dominance	152
5.3.5	Efficient Sets and the Portfolio Choice Problem	154
5.3.6	Return versus Payoff	154

5.4	Probability Metrics and Stochastic Dominance	157
5.5	Summary	161
5.6	Technical Appendix	161
5.6.1	The Axioms of Choice	161
5.6.2	Stochastic Dominance Relations of Order n	163
5.6.3	Return versus Payoff and Stochastic Dominance	164
5.6.4	Other Stochastic Dominance Relations	166

CHAPTER 6

Risk and Uncertainty

171

6.1	Introduction	171
6.2	Measures of Dispersion	174
6.2.1	Standard Deviation	174
6.2.2	Mean Absolute Deviation	176
6.2.3	Semistandard Deviation	177
6.2.4	Axiomatic Description	178
6.2.5	Deviation Measures	179
6.3	Probability Metrics and Dispersion Measures	180
6.4	Measures of Risk	181
6.4.1	Value-at-Risk	182
6.4.2	Computing Portfolio VaR in Practice	186
6.4.3	Backtesting of VaR	192
6.4.4	Coherent Risk Measures	194
6.5	Risk Measures and Dispersion Measures	198
6.6	Risk Measures and Stochastic Orders	199
6.7	Summary	200
6.8	Technical Appendix	201
6.8.1	Convex Risk Measures	201
6.8.2	Probability Metrics and Deviation Measures	202

CHAPTER 7

Average Value-at-Risk

207

7.1	Introduction	207
7.2	Average Value-at-Risk	208
7.3	AVaR Estimation from a Sample	214
7.4	Computing Portfolio AVaR in Practice	216
7.4.1	The Multivariate Normal Assumption	216
7.4.2	The Historical Method	217
7.4.3	The Hybrid Method	217
7.4.4	The Monte Carlo Method	218
7.5	Backtesting of AVaR	220

7.6	Spectral Risk Measures	222
7.7	Risk Measures and Probability Metrics	224
7.8	Summary	227
7.9	Technical Appendix	227
7.9.1	Characteristics of Conditional Loss Distributions	228
7.9.2	Higher-Order AVaR	230
7.9.3	The Minimization Formula for AVaR	232
7.9.4	AVaR for Stable Distributions	235
7.9.5	ETL versus AVaR	236
7.9.6	Remarks on Spectral Risk Measures	241

CHAPTER 8

Optimal Portfolios

245

8.1	Introduction	245
8.2	Mean-Variance Analysis	247
8.2.1	Mean-Variance Optimization Problems	247
8.2.2	The Mean-Variance Efficient Frontier	251
8.2.3	Mean-Variance Analysis and SSD	254
8.2.4	Adding a Risk-Free Asset	256
8.3	Mean-Risk Analysis	258
8.3.1	Mean-Risk Optimization Problems	259
8.3.2	The Mean-Risk Efficient Frontier	262
8.3.3	Mean-Risk Analysis and SSD	266
8.3.4	Risk versus Dispersion Measures	267
8.4	Summary	274
8.5	Technical Appendix	274
8.5.1	Types of Constraints	274
8.5.2	Quadratic Approximations to Utility Functions	276
8.5.3	Solving Mean-Variance Problems in Practice	278
8.5.4	Solving Mean-Risk Problems in Practice	279
8.5.5	Reward-Risk Analysis	281

CHAPTER 9

Benchmark Tracking Problems

287

9.1	Introduction	287
9.2	The Tracking Error Problem	288
9.3	Relation to Probability Metrics	292
9.4	Examples of r.d. Metrics	296
9.5	Numerical Example	300
9.6	Summary	304

9.7	Technical Appendix	304
9.7.1	Deviation Measures and r.d. Metrics	305
9.7.2	Remarks on the Axioms	305
9.7.3	Minimal r.d. Metrics	307
9.7.4	Limit Cases of $\mathcal{L}_p^*(X, Y)$ and $\Theta_p^*(X, Y)$	310
9.7.5	Computing r.d. Metrics in Practice	311
CHAPTER 10		
Performance Measures		317
10.1	Introduction	317
10.2	Reward-to-Risk Ratios	318
10.2.1	RR Ratios and the Efficient Portfolios	320
10.2.2	Limitations in the Application of Reward-to-Risk Ratios	324
10.2.3	The STARR	325
10.2.4	The Sortino Ratio	329
10.2.5	The Sortino-Satchell Ratio	330
10.2.6	A One-Sided Variability Ratio	331
10.2.7	The Rachev Ratio	332
10.3	Reward-to-Variability Ratios	333
10.3.1	RV Ratios and the Efficient Portfolios	335
10.3.2	The Sharpe Ratio	337
10.3.3	The Capital Market Line and the Sharpe Ratio	340
10.4	Summary	343
10.5	Technical Appendix	343
10.5.1	Extensions of STARR	343
10.5.2	Quasiconcave Performance Measures	345
10.5.3	The Capital Market Line and Quasiconcave Ratios	353
10.5.4	Nonquasiconcave Performance Measures	356
10.5.5	Probability Metrics and Performance Measures	357
Index		361

Preface

Modern portfolio theory, as pioneered in the 1950s by Harry Markowitz, is well adopted by the financial community. In spite of the fundamental shortcomings of mean-variance analysis, it remains a basic tool in the industry.

Since the 1990s, significant progress has been made in developing the concept of a risk measure from both a theoretical and a practical viewpoint. This notion has evolved into a materially different form from the original idea behind mean-variance analysis. As a consequence, the distinction between risk and uncertainty, which translates into a distinction between a risk measure and a dispersion measure, offers a new way of looking at the problem of optimal portfolio selection.

As concepts develop, other tools become appropriate to exploring evolved ideas than existing techniques. In applied finance, these tools are being imported from mathematics. That said, we believe that probability metrics, which is a field in probability theory, will turn out to be well-positioned for the study and further development of the quantitative aspects of risk and uncertainty. Going one step further, we make a parallel. In the theory of probability metrics, there exists a concept known as an *ideal probability metric*. This is a quantity best suited for the study of a given approximation problem in probability or stochastic processes. We believe that the ideas behind this concept can be borrowed and applied in the field of asset management to construct an *ideal risk measure* that would be ideal for a given optimal portfolio selection problem.

The development of probability metrics as a branch of probability theory started in the 1950s, even though its basic ideas were used during the first half of the 20th century. Its application to problems is connected with this fundamental question: “Is the proposed stochastic model a satisfactory approximation to the real model and, if so, within what limits?” In finance, we assume a stochastic model for asset return distributions and, in order to estimate portfolio risk, we sample from the fitted distribution. Then we use the generated simulations to evaluate the portfolio positions and, finally, to calculate portfolio risk. In this context, there are two issues arising on two different levels. First, the assumed stochastic model should be close to the empirical data. That is, we need a realistic model in the first place. Second, the generated scenarios should be sufficiently many in order to represent a