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V. G. Turaev

Quantum Invariants of Knots and 3-Manifolds

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Dedicated to my parents

Contents

Introduction

In the last decade we have witnessed the birth of a fascinating new mathematical theory. It is often called by algebraists the theory of quantum groups and by topologists quantum topology. These terms, however, seem to be too restrictive and do not convey the breadth of this new domain which is closely related to the theory of von Neumann algebras, the theory of Hopf algebras, the theory of representations of semisimple Lie algebras, the topology of knots, etc. The most spectacular achievements in this theory are centered around quantum groups and invariants of knots and 3-dimensional manifolds.

The whole theory has been, to a great extent, inspired by ideas that arose in theoretical physics. Among the relevant areas of physics are the theory of exactly solvable models of statistical mechanics, the quantum inverse scattering method, the quantum theory of angular momentum, 2-dimensional conformal field theory, etc. The development of this subject shows once more that physics and mathematics intercommunicate and influence each other to the profit of both disciplines.

Three major events have marked the history of this theory. A powerful original impetus was the introduction of a new polynomial invariant of classical knots and links by V. Jones (1984). This discovery drastically changed the scenery of knot theory. The Jones polynomial paved the way for an intervention of von Neumann algebras, Lie algebras, and physics into the world of knots and 3-manifolds.

The second event was the introduction by V. Drinfel'd and M. Jimbo (1985) of quantum groups which may roughly be described as 1-parameter deformations of semisimple complex Lie algebras. Quantum groups and their representation theory form the algebraic basis and environment for this subject. Note that quantum groups emerged as an algebraic formalism for physicists' ideas, specifically, from the work of the Leningrad school of mathematical physics directed by L. Faddeev.

In 1988 E. Witten invented the notion of a topological quantum field theory and outlined a fascinating picture of such a theory in three dimensions. This picture includes an interpretation of the Jones polynomial as a path integral and relates the Jones polynomial to a 2-dimensional modular functor arising in conformal field theory. It seems that at the moment of writing (beginning of 1994), Witten's approach based on path integrals has not yet been justified mathematically. Witten's conjecture on the existence of non-trivial 3-dimensional TQFT's has served as a major source of inspiration for the research in this area.

The development of the subject (in its topological part) has been strongly influenced by the works of M. Atiyah, A. Joyal and R. Street, L. Kauffman,

A. Kirillov and N. Reshetikhin, G. Moore and N. Seiberg, N. Reshetikhin and V. Turaev, G. Segal, V. Turaev and O. Viro, and others (see References). Although this theory is very young, the number of relevant papers is overwhelming. We do not attempt to give a comprehensive history of the subject and confine ourselves to sketchy historical remarks in the chapter notes.

In this monograph we focus our attention on the topological aspects of the theory. Our goal is the construction and study of invariants of knots and 3-manifolds. There are several possible approaches to these invariants, based on Chern-Simons field theory, 2-dimensional conformal field theory, and quantum groups. We shall follow the last approach. The fundamental idea is to derive invariants of knots and 3-manifolds from algebraic objects which formalize the properties of modules over quantum groups at roots of unity. This approach allows a rigorous mathematical treatment of a number of ideas considered in theoretical physics.

This monograph is addressed to mathematicians and physicists with a knowledge of basic algebra and topology. We do not assume that the reader is acquainted with the theory of quantum groups or with the relevant chapters of mathematical physics.

Besides an exposition of the material available in published papers, this monograph presents new results of the author, which appear here for the first time. Indications to this effect and priority references are given in the chapter notes.

The fundamental notions discussed in the monograph are those of modular category, modular functor, and topological quantum field theory (TQFT). The mathematical content of these notions may be outlined as follows.

Modular categories are tensor categories with certain additional algebraic structures (braiding, twist) and properties of semisimplicity and finiteness. The notions of braiding and twist arise naturally from the study of the commutativity of the tensor product. Semisimplicity means that all objects of the category may be decomposed into "simple" objects which play the role of irreducible modules in representation theory. Finiteness means that such a decomposition can be performed using only a finite stock of simple objects.

The use of categories should not frighten the reader unaccustomed to the abstract theory of categories. Modular categories are defined in algebraic terms and have a purely algebraic nature. Still, if the reader wants to avoid the language of categories, he may think of the objects of a modular category as finite dimensional modules over a Hopf algebra.

Modular functors relate topology to algebra and are reminiscent of homology. A modular functor associates projective modules over a fixed commutative ring K to certain "nice" topological spaces. When we speak of an n-dimensional modular functor, the role of "nice" spaces is played by closed n-dimensional manifolds (possibly with additional structures like orientation, smooth structure, etc.). An

n-dimensional modular functor $\mathcal{T}$ assigns to a closed n-manifold (with a certain additional structure) Σ, a projective K-module $\mathcal{T}(\Sigma)$, and assigns to a homeomorphism of n-manifolds (preserving the additional structure), an isomorphism of the corresponding modules. The module $\mathcal{T}(\Sigma)$ is called the module of states of Σ. These modules should satisfy a few axioms including multiplicativity with respect to disjoint union: $\mathcal{T}(\Sigma \amalg \Sigma') = \mathcal{T}(\Sigma) \otimes_K \mathcal{T}(\Sigma')$. It is convenient to regard the empty space as an n-manifold and to require that $\mathcal{T}(\emptyset) = K$.

A modular functor may sometimes be extended to a topological quantum field theory (TQFT), which associates homomorphisms of modules of states to cobordisms ("spacetimes"). More precisely, an $(n+1)$-dimensional TQFT is formed by an n-dimensional modular functor $\mathcal{T}$ and an operator invariant of $(n+1)$-cobordisms τ. By an $(n+1)$-cobordism, we mean a compact $(n+1)$-manifold M whose boundary is a disjoint union of two closed n-manifolds $\partial_- M$, $\partial_+ M$ called the bottom base and the top base of M. The operator invariant τ assigns to such a cobordism M a homomorphism

$$\tau(M) : \mathcal{T}(\partial_- M) \to \mathcal{T}(\partial_+ M).$$

This homomorphism should be invariant under homeomorphisms of cobordisms and multiplicative with respect to disjoint union of cobordisms. Moreover, τ should be compatible with gluings of cobordisms along their bases: if a cobordism M is obtained by gluing two cobordisms M_1 and M_2 along their common base $\partial_+(M_1) = \partial_-(M_2)$ then

$$\tau(M) = k\,\tau(M_2) \circ \tau(M_1) : \mathcal{T}(\partial_-(M_1)) \to \mathcal{T}(\partial_+(M_2))$$

where $k \in K$ is a scalar factor depending on M, M_1, M_2. The factor k is called the anomaly of the gluing. The most interesting TQFT's are those which have no gluing anomalies in the sense that for any gluing, $k = 1$. Such TQFT's are said to be anomaly-free.

In particular, a closed $(n+1)$-manifold M may be regarded as a cobordism with empty bases. The operator $\tau(M)$ acts in $\mathcal{T}(\emptyset) = K$ as multiplication by an element of K. This element is the "quantum" invariant of M provided by the TQFT $(\mathcal{T}, \tau)$. It is denoted also by $\tau(M)$.

We note that to speak of a TQFT $(\mathcal{T}, \tau)$, it is necessary to specify the class of spaces and cobordisms subject to the application of $\mathcal{T}$ and τ.

In this monograph we shall consider 2-dimensional modular functors and 3-dimensional topological quantum field theories. Our main result asserts that every modular category gives rise to an anomaly-free 3-dimensional TQFT:

$$\text{modular category} \quad \mapsto \quad \text{3-dimensional TQFT}.$$

In particular, every modular category gives rise to a 2-dimensional modular functor:

$$\text{modular category} \quad \mapsto \quad \text{2-dimensional modular functor.}$$

The 2-dimensional modular functor $\mathcal{T}_{\mathcal{V}}$, derived from a modular category $\mathcal{V}$, applies to closed oriented surfaces with a distinguished Lagrangian subspace in 1-homologies and a finite (possibly empty) set of marked points. A point of a surface is marked if it is endowed with a non-zero tangent vector, a sign ± 1, and an object of $\mathcal{V}$; this object of $\mathcal{V}$ is regarded as the "color" of the point. The modular functor $\mathcal{T}_{\mathcal{V}}$ has a number of interesting properties including nice behavior with respect to cutting surfaces out along simple closed curves. Borrowing terminology from conformal field theory, we say that $\mathcal{T}_{\mathcal{V}}$ is a rational 2-dimensional modular functor.

We shall show that the modular category $\mathcal{V}$ can be reconstructed from the corresponding modular functor $\mathcal{T}_{\mathcal{V}}$. This deep fact shows that the notions of modular category and rational 2-dimensional modular functor are essentially equivalent; they are two sides of the same coin formulated in algebraic and geometric terms:

$$\text{modular category} \quad \Longleftrightarrow \quad \text{rational 2-dimensional modular functor.}$$

The operator invariant τ, derived from a modular category $\mathcal{V}$, applies to compact oriented 3-cobordisms whose bases are closed oriented surfaces with the additional structure as above. The cobordisms may contain colored framed oriented knots, links, or graphs which meet the bases of the cobordism along the marked points. (A link is colored if each of its components is endowed with an object of $\mathcal{V}$. A link is framed if it is endowed with a non-singular normal vector field in the ambient 3-manifold.) For closed oriented 3-manifolds and for colored framed oriented links in such 3-manifolds, this yields numerical invariants. These are the "quantum" invariants of links and 3-manifolds derived from $\mathcal{V}$. Under a special choice of $\mathcal{V}$ and a special choice of colors, we recover the Jones polynomial of links in the 3-sphere S^3 or, more precisely, the value of this polynomial at a complex root of unity.

An especially important class of 3-dimensional TQFT's is formed by so-called unitary TQFT's with ground ring $K = \mathbb{C}$. In these TQFT's, the modules of states of surfaces are endowed with positive definite Hermitian forms. The corresponding algebraic notion is the one of a unitary modular category. We show that such categories give rise to unitary TQFT's:

$$\text{unitary modular category} \quad \mapsto \quad \text{unitary 3-dimensional TQFT.}$$

Unitary 3-dimensional TQFT's are considerably more sensitive to the topology of 3-manifolds than general TQFT's. They can be used to estimate certain classical numerical invariants of knots and 3-manifolds.

To sum up, we start with a purely algebraic object (a modular category) and build a topological theory of modules of states of surfaces and operator invariants of 3-cobordisms. This construction reveals an algebraic background to 2-dimensional modular functors and 3-dimensional TQFT's. It is precisely because there are non-trivial modular categories, that there exist non-trivial 3-dimensional TQFT's.

The construction of a 3-dimensional TQFT from a modular category $\mathcal{V}$ is the central result of Part I of the book. We give here a brief overview of this construction.

The construction proceeds in several steps. First, we define an isotopy invariant F of colored framed oriented links in Euclidean space $\mathbb{R}^3$. The invariant F takes values in the commutative ring $K = \mathrm{Hom}_{\mathcal{V}}(\mathbb{1}, \mathbb{1})$, where $\mathbb{1}$ is the unit object of $\mathcal{V}$. The main idea in the definition of F is to dissect every link $L \subset \mathbb{R}^3$ into elementary "atoms". We first deform L in $\mathbb{R}^3$ so that its normal vector field is given everywhere by the vector (0,0,1). Then we draw the orthogonal projection of L in the plane $\mathbb{R}^2 = \mathbb{R}^2 \times 0$ taking into account overcrossings and undercrossings. The resulting plane picture is called the diagram of L. It is convenient to think that the diagram is drawn on graph paper. Stretching the diagram in the vertical direction, if necessary, we may arrange that each small square of the paper contains either one vertical line of the diagram, an X-like crossing of two lines, a cap-like arc $\cap$, or a cup-like arc $\cup$. These are the atoms of the diagram. We use the algebraic structures in $\mathcal{V}$ and the colors of link components to assign to each atom a morphism in $\mathcal{V}$. Using the composition and tensor product in $\mathcal{V}$, we combine the morphisms corresponding to the atoms of the diagram into a single morphism $F(L) : \mathbb{1} \to \mathbb{1}$. To verify independence of $F(L) \in K$ on the choice of the diagram, we appeal to the fact that any two diagrams of the same link may be related by Reidemeister moves and local moves changing the position of the diagram with respect to the squares of graph paper.

The invariant F may be generalized to an isotopy invariant of colored graphs in $\mathbb{R}^3$. By a coloring of a graph, we mean a function which assigns to every edge an object of $\mathcal{V}$ and to every vertex a morphism in $\mathcal{V}$. The morphism assigned to a vertex should be an intertwiner between the objects of $\mathcal{V}$ sitting on the edges incident to this vertex. As in the case of links we need a kind of framing for graphs, specifically, we consider ribbon graphs whose edges and vertices are narrow ribbons and small rectangles.

Note that this part of the theory does not use semisimplicity and finiteness of $\mathcal{V}$. The invariant F can be defined for links and ribbon graphs in $\mathbb{R}^3$ colored over arbitrary tensor categories with braiding and twist. Such categories are called ribbon categories.

Next we define a topological invariant $\tau(M) = \tau_{\mathcal{V}}(M) \in K$ for every closed oriented 3-manifold M. Present M as the result of surgery on the 3-sphere $S^3 = \mathbb{R}^3 \cup \{\infty\}$ along a framed link $L \subset \mathbb{R}^3$. Orient L in an arbitrary way and vary the colors of the components of L in the finite family of simple objects of $\mathcal{V}$ appearing in the definition of a modular category. This gives a finite family of colored (framed oriented) links in $\mathbb{R}^3$ with the same underlying link L. We define $\tau(M)$ to be a certain weighted sum of the corresponding invariants $F \in K$. To verify independence on the choice of L, we use the Kirby calculus of links allowing us to relate any two choices of L by a sequence of local geometric transformations.

The invariant $\tau(M) \in K$ generalizes to an invariant $\tau(M, \Omega) \in K$ where M is a closed oriented 3-manifold and Ω is a colored ribbon graph in M.

At the third step we define an auxiliary 3-dimensional TQFT that applies to parametrized surfaces and 3-cobordisms with parametrized bases. A surface is parametrized if it is provided with a homeomorphism onto the standard closed surface of the same genus bounding a standard unknotted handlebody in $\mathbb{R}^3$. Let M be an oriented 3-cobordism with parametrized boundary (this means that all components of ∂M are parametrized). Consider first the case where $\partial_+ M = \emptyset$ and $\Sigma = \partial_- M$ is connected. Gluing the standard handlebody to M along the parametrization of Σ yields a closed 3-manifold $\tilde{M}$. We consider a certain canonical ribbon graph R in the standard handlebody in $\mathbb{R}^3$ lying there as a kind of core and having only one vertex. Under the gluing used above, R embeds in $\tilde{M}$. We color the edges of R with arbitrary objects from the finite family of simple objects appearing in the definition of $\mathcal{V}$. Coloring the vertex of R with an intertwiner we obtain a colored ribbon graph $\tilde{R} \subset \tilde{M}$. Denote by $\mathcal{T}(\Sigma)$ the K-module formally generated by such colorings of R. We can regard $\tau(\tilde{M}, \tilde{R}) \in K$ as a linear functional $\mathcal{T}(\Sigma) \to K$. This is the operator $\tau(M)$. The case of a 3-cobordism with non-connected boundary is treated similarly: we glue standard handlebodies (with the standard ribbon graphs inside) to all the components of ∂M and apply τ as above. This yields a linear functional on the tensor product $\otimes_i \mathcal{T}(\Sigma_i)$ where Σ_i runs over the components of ∂M. Such a functional may be rewritten as a linear operator $\mathcal{T}(\partial_- M) \to \mathcal{T}(\partial_+ M)$.

The next step is to define the action of surface homeomorphisms in the modules of states and to replace parametrizations of surfaces with a less rigid structure. The study of homeomorphisms may be reduced to a study of 3-cobordisms with parametrized bases. Namely, if Σ is a standard surface then any homeomorphism $f : \Sigma \to \Sigma$ gives rise to the 3-cobordism $(\Sigma \times [0, 1], \Sigma \times 0, \Sigma \times 1)$ whose bottom base is parametrized via f and whose top base is parametrized via id_Σ. The operator invariant τ of this cobordism yields an action of f in $\mathcal{T}(\Sigma)$. This gives a projective linear action of the group $\mathrm{Homeo}(\Sigma)$ on $\mathcal{T}(\Sigma)$. The corresponding 2-cocycle is computed in terms of Maslov indices of Lagrangian spaces in $H_1(\Sigma; \mathbb{R})$. This computation implies that the module $\mathcal{T}(\Sigma)$ does not depend on the choice of parametrization, but rather depends on the Lagrangian space in $H_1(\Sigma; \mathbb{R})$ determined by this parametrization. This fact allows us to define a TQFT based on closed oriented surfaces endowed with a distinguished Lagrangian space in 1-homologies and on compact oriented 3-cobordisms between such surfaces. Finally, we show how to modify this TQFT in order to kill its gluing anomalies.

The definition of the quantum invariant $\tau(M) = \tau_{\mathcal{V}}(M)$ of a closed oriented 3-manifold M is based on an elaborate reduction to link diagrams. It would be most important to compute $\tau(M)$ in intrinsic terms, i.e., directly from M rather than from a link diagram. In Part II of the book we evaluate in intrinsic terms the product $\tau(M)\,\tau(-M)$ where $-M$ denotes the same manifold M with the opposite orientation. More precisely, we compute $\tau(M)\,\tau(-M)$ as a state sum on a triangulation of M. In the case of a unitary modular category,

$$\tau(M)\,\tau(-M) = |\tau(M)|^2 \in \mathbb{R}$$

so that we obtain the absolute value of $\tau(M)$ as the square root of a state sum on a triangulation of M.

The algebraic ingredients of the state sum in question are so-called $6j$-symbols associated to $\mathcal{V}$. The $6j$-symbols associated to the Lie algebra $sl_2(\mathbb{C})$ are well known in the quantum theory of angular momentum. These symbols are complex numbers depending on 6 integer indices. We define more general $6j$-symbols associated to a modular category $\mathcal{V}$ satisfying a minor technical condition of unimodality. In the context of modular categories, each $6j$-symbol is a tensor in 4 variables running over so-called multiplicity modules. The $6j$-symbols are numerated by tuples of 6 indices running over the set of distinguished simple objects of $\mathcal{V}$. The system of $6j$-symbols describes the associativity of the tensor product in $\mathcal{V}$ in terms of multiplicity modules. A study of $6j$-symbols inevitably appeals to geometric images. In particular, the appearance of the numbers 4 and 6 has a simple geometric interpretation: we should think of the 6 indices mentioned above as sitting on the edges of a tetrahedron while the 4 multiplicity modules sit on its 2-faces. This interpretation is a key to applications of $6j$-symbols in 3-dimensional topology.

We define a state sum on a triangulated closed 3-manifold M as follows. Color the edges of the triangulation with distinguished simple objects of $\mathcal{V}$. Associate to each tetrahedron of the triangulation the $6j$-symbol determined by the colors of its 6 edges. This $6j$-symbol lies in the tensor product of 4 multiplicity modules associated to the faces of the tetrahedron. Every 2-face of the triangulation is incident to two tetrahedra and contributes dual multiplicity modules to the corresponding tensor products. We consider the tensor product of $6j$-symbols associated to all tetrahedra of the triangulation and contract it along the dualities determined by 2-faces. This gives an element of the ground ring K corresponding to the chosen coloring. We sum up these elements (with certain coefficients) over all colorings. The sum does not depend on the choice of triangulation and yields a homeomorphism invariant $|M| \in K$ of M. It turns out that for oriented M, we have

$$|M| = \tau(M)\,\tau(-M).$$

Similar state sums on 3-manifolds with boundary give rise to a so-called simplicial TQFT based on closed surfaces and compact 3-manifolds (without additional structures). The equality $|M| = \tau(M)\tau(-M)$ for closed oriented 3-manifolds generalizes to a splitting theorem for this simplicial TQFT.

The proof of the formula $|M| = \tau(M)\tau(-M)$ is based on a computation of $\tau(M)$ inside an arbitrary compact oriented piecewise-linear 4-manifold bounded by M. This result, interesting in itself, gives a 4-dimensional perspective to quantum invariants of 3-manifolds. The computation in question involves the fundamental notion of shadows of 4-manifolds. Shadows are purely topological objects intimately related to $6j$-symbols. The theory of shadows was, to a great extent, stimulated by a study of 3-dimensional TQFT's.

The idea underlying the definition of shadows is to consider 2-dimensional polyhedra whose 2-strata are provided with numbers. We shall consider only so-called simple 2-polyhedra. Every simple 2-polyhedron naturally decomposes into a disjoint union of vertices, 1-strata (edges and circles), and 2-strata. We say that a simple 2-polyhedron is shadowed if each of its 2-strata is endowed with an integer or half-integer, called the gleam of this 2-stratum. We define three local transformations of shadowed 2-polyhedra (shadow moves). A shadow is a shadowed 2-polyhedron regarded up to these moves.

Being 2-dimensional, shadows share many properties with surfaces. For instance, there is a natural notion of summation of shadows similar to the connected summation of surfaces. It is more surprising that shadows share a number of important properties of 3-manifolds and 4-manifolds. In analogy with 3-manifolds they may serve as ambient spaces of knots and links. In analogy with 4-manifolds they possess a symmetric bilinear form in 2-homologies. Imitating surgery and cobordism for 4-manifolds, we define surgery and cobordism for shadows.

Shadows arise naturally in 4-dimensional topology. Every compact oriented piecewise-linear 4-manifold W (possibly with boundary) gives rise to a shadow $\mathrm{sh}(W)$. To define $\mathrm{sh}(W)$, we consider a simple 2-skeleton of W and provide every 2-stratum with its self-intersection number in W. The resulting shadowed polyhedron considered up to shadow moves and so-called stabilization does not depend on the choice of the 2-skeleton. In technical terms, $\mathrm{sh}(W)$ is a stable integer shadow. Thus, we have an arrow

$$\text{compact oriented PL 4-manifolds} \mapsto \text{stable integer shadows.}$$

It should be emphasized that this part of the theory is purely topological and does not involve tensor categories.

Every modular category $\mathcal{V}$ gives rise to an invariant of stable shadows. It is obtained via a state sum on shadowed 2-polyhedra. The algebraic ingredients of this state sum are the $6j$-symbols associated to $\mathcal{V}$. This yields a mapping

$$\text{stable integer shadows} \xrightarrow{\text{state sum}} K = \mathrm{Hom}_{\mathcal{V}}(\mathbb{1}, \mathbb{1}).$$

Composing these arrows we obtain a K-valued invariant of compact oriented PL 4-manifolds. By a miracle, this invariant of a 4-manifold W depends only on ∂W and coincides with $\tau(\partial W)$. This gives a computation of $\tau(\partial W)$ inside W.

The discussion above naturally raises the problem of existence of modular categories. These categories are quite delicate algebraic objects. Although there are elementary examples of modular categories, it is by no means obvious that there exist modular categories leading to deep topological theories. The source of interesting modular categories is the theory of representations of quantum groups at roots of unity. The quantum group $U_q(\mathfrak{g})$ is a Hopf algebra over $\mathbb{C}$ obtained by a 1-parameter deformation of the universal enveloping algebra of a simple Lie algebra $\mathfrak{g}$. The finite dimensional modules over $U_q(\mathfrak{g})$ form a semisimple tensor category with braiding and twist. To achieve finiteness, we take the deformation parameter q to be a complex root of unity. This leads to a loss of semisimplicity which is regained under the passage to a quotient category. If $\mathfrak{g}$ belongs to the series A, B, C, D and the order of the root of unity q is even and sufficiently big then we obtain a modular category with ground ring $\mathbb{C}$:

$$\text{quantum group at a root of 1} \quad \mapsto \quad \text{modular category.}$$

Similar constructions may be applied to exceptional simple Lie algebras, although some details are yet to be worked out. It is remarkable that for $q = 1$ we have the classical theory of representations of a simple Lie algebra while for non-trivial complex roots of unity we obtain modular categories.

Summing up, we may say that the simple Lie algebras of the series A, B, C, D give rise to 3-dimensional TQFT's via the q-deformation, the theory of representations, and the theory of modular categories. The resulting 3-dimensional TQFT's are highly non-trivial from the topological point of view. They yield new invariants of 3-manifolds and knots including the Jones polynomial (which is obtained from $\mathfrak{g} = sl_2(\mathbb{C})$) and its generalizations.

At earlier stages in the theory of quantum 3-manifold invariants, Hopf algebras and quantum groups played the role of basic algebraic objects, i.e., the role of modular categories in our present approach. It is in this book that we switch to categories. Although the language of categories is more general and more simple, it is instructive to keep in mind its algebraic origins.

There is a dual approach to the modular categories derived from the quantum groups $U_q(sl_n(\mathbb{C}))$ at roots of unity. The Weyl duality between representations of $U_q(sl_n(\mathbb{C}))$ and representations of Hecke algebras suggests that one should study the categories whose objects are idempotents of Hecke algebras. We shall treat the simplest but most important case, $n = 2$. In this case instead of Hecke algebras we may consider their quotients, the Temperley-Lieb algebras. A study of idempotents in the Temperley-Lieb algebras together with the skein theory of tangles gives a construction of modular categories. This construction is elementary and self-contained. It completely avoids the theory of quantum groups but yields the same modular categories as the representation theory of $U_q(sl_2(\mathbb{C}))$ at roots of unity.

Until now main efforts have been spent to construct 2-dimensional modular functors and 3-dimensional TQFT's. Little is known about their role in low-dimensional topology. Relationships between the TQFT's and the classical invariants of 3-manifolds, for instance, the fundamental group are poorly understood. Topological properties and applications of the modular functors derived from quantum groups are yet to be studied. The most important problem is to relate the 3-dimensional TQFT's to the Donaldson invariants of 4-manifolds and the Floer homologies of 3-manifolds. (Very interesting results in this direction have been recently obtained by H. Murakami.)

The book consists of three parts. Part I (Chapters I - V) is concerned with the construction of a 2-dimensional modular functor and 3-dimensional TQFT from a modular category. Part II (Chapters VI - X) deals with $6j$-symbols, shadows, and state sums on shadows and 3-manifolds. Part III (Chapters XI, XII) is concerned with constructions of modular categories.

It is possible but not at all necessary to read the chapters in their linear order. The reader may start with Chapter III or with Chapters VIII, IX which are independent of the previous material. It is also possible to start with Part III which is almost independent of Parts I and II, one needs only to be acquainted with the definitions of ribbon, modular, semisimple, Hermitian, and unitary categories given in Section I.1 (i.e., Section 1 of Chapter I) and Sections II.1, II.4, II.5.

The interdependence of the chapters is presented in the following diagram. An arrow from A to B indicates that the definitions and results of Chapter A are essential for Chapter B. Weak dependence of chapters is indicated by dotted arrows.

$$
\begin{tikzcd}
 & \mathrm{I} \arrow[d] & \mathrm{III} \arrow[d] & & \mathrm{VIII} \arrow[d] \\
\mathrm{XI} & \mathrm{II} \arrow[l, dotted] \arrow[r] \arrow[d] \arrow[dl, dotted] & \mathrm{IV} \arrow[r] \arrow[d, dotted] \arrow[dr, dotted] & \mathrm{V} & \mathrm{IX} \arrow[dl] \\
\mathrm{XII} & \mathrm{VI} \arrow[r] & \mathrm{VII} \arrow[r] & \mathrm{X} &
\end{tikzcd}
$$

The content of the chapters should be clear from the headings. The following remarks give more directions to the reader.

Chapter I starts off with ribbon categories and invariants of colored framed graphs and links in Euclidean 3-space. The relevant definitions and results, given in the first two sections of Chapter I, will be used throughout the book. They contain the seeds of main ideas of the theory. Sections I.3 and I.4 are concerned with the proof of Theorem I.2.5 and may be skipped without much loss.

Chapter II starts with two fundamental sections. In Section II.1 we introduce modular categories which are the main algebraic objects of the monograph. In Section II.2 we introduce the invariant τ of closed oriented 3-manifolds. In Section II.3 we prove that τ is well defined. The ideas of the proof are used in the same section to construct a projective linear action of the group $SL(2, \mathbb{Z})$. This action does not play an important role in the book, rather it serves as a precursor for the actions of modular groups of surfaces on the modules of states introduced in Chapter IV. In Section II.4 we define semisimple ribbon categories and establish an analogue of the Verlinde-Moore-Seiberg formula known in conformal field theory. Section II.5 is concerned with Hermitian and unitary modular categories.

Chapter III deals with axiomatic foundations of topological quantum field theory. It is remarkable that even in a completely abstract set up, we can establish meaningful theorems which prove to be useful in the context of 3-dimensional TQFT's. The most important part of Chapter III is the first section where we give an axiomatic definition of modular functors and TQFT's. The language introduced in Section III.1 will be used systematically in Chapter IV. In Section III.2 we establish a few fundamental properties of TQFT's. In Section III.3 we introduce the important notion of a non-degenerate TQFT and establish sufficient conditions for isomorphism of non-degenerate anomaly-free TQFT's. Section III.5 deals with Hermitian and unitary TQFT's, this study will be continued in the 3-dimensional setting at the end of Chapter IV. Sections III.4 and III.6 are more or less isolated from the rest of the book; they deal with the abstract notion of a quantum invariant of topological spaces and a general method of killing the gluing anomalies of a TQFT.

In Chapter IV we construct the 3-dimensional TQFT associated to a modular category. It is crucial for the reader to get through Section IV.1, where we define the 3-dimensional TQFT for 3-cobordisms with parametrized boundary. Section IV.2 provides the proofs for Section IV.1; the geometric technique of Section IV.2 is probably one of the most difficult in the book. However, this technique is used only a few times in the remaining part of Chapter IV and in Chapter V. Section IV.3 is purely algebraic and independent of all previous sections. It provides generalities on Lagrangian relations and Maslov indices. In Sections IV.4 - IV.6 we show how to renormalize the TQFT introduced in Section IV.1 in order to replace parametrizations of surfaces with Lagrangian spaces in 1-homologies. The 3-dimensional TQFT $(\mathcal{T}^e, \tau^e)$, constructed in Section IV.6 and further studied in Section IV.7, is quite suitable for computations and applications. This TQFT has anomalies which are killed in Sections IV.8 and IV.9 in two different ways. The anomaly-free TQFT constructed in Section IV.9 is the final product of Chapter IV. In Sections IV.10 and IV.11 we show that the TQFT's derived from Hermitian (resp. unitary) modular categories are themselves Hermitian (resp. unitary). In the purely algebraic Section IV.12 we introduce the Verlinde algebra of a modular category and use it to compute the dimension of the module of states of a surface.

The results of Chapter IV shall be used in Sections V.4, V.5, VII.4, and X.8.

Chapter V is devoted to a detailed analysis of the 2-dimensional modular functors (2-DMF's) arising from modular categories. In Section V.1 we give an axiomatic definition of 2-DMF's and rational 2-DMF's independent of all previous material. In Section V.2 we show that each (rational) 2-DMF gives rise to a (modular) ribbon category. In Section V.3 we introduce the more subtle notion of a weak rational 2-DMF. In Sections V.4 and V.5 we show that the constructions of Sections IV.1 - IV.6, being properly reformulated, yield a weak rational 2-DMF.

Chapter VI deals with $6j$-symbols associated to a modular category. The most important part of this chapter is Section VI.5, where we use the invariants of ribbon graphs introduced in Chapter I to define so-called normalized $6j$-symbols. They should be contrasted with the more simple-minded $6j$-symbols defined in Section VI.1 in a direct algebraic way. The approach of Section VI.1 generalizes the standard definition of $6j$-symbols but does not go far enough. The fundamental advantage of normalized $6j$-symbols is their tetrahedral symmetry. Three intermediate sections (Sections VI.2 - VI.4) prepare different kinds of preliminary material necessary to define the normalized $6j$-symbols.

In the first section of Chapter VII we use $6j$-symbols to define state sums on triangulated 3-manifolds. Independence on the choice of triangulation is shown in Section VII.2. Simplicial 3-dimensional TQFT is introduced in Section VII.3. Finally, in Section VII.4 we state the main theorems of Part II; they relate the theory developed in Part I to the state sum invariants of closed 3-manifolds and simplicial TQFT's.

Chapters VIII and IX are purely topological. In Chapter VIII we discuss the general theory of shadows. In Chapter IX we consider shadows of 4-manifolds, 3-manifolds, and links in 3-manifolds. The most important sections of these two chapters are Sections VIII.1 and IX.1 where we define (abstract) shadows and shadows of 4-manifolds. The reader willing to simplify his way towards Chapter X may read Sections VIII.1, VIII.2.1, VIII.2.2, VIII.6, IX.1 and then proceed to Chapter X coming back to Chapters VIII and IX when necessary.

In Chapter X we combine all the ideas of the previous chapters. We start with state sums on shadowed 2-polyhedra based on normalized $6j$-symbols (Section X.1) and show their invariance under shadow moves (Section X.2). In Section X.3 we interpret the invariants of closed 3-manifolds $\tau(M)$ and $|M|$ introduced in Chapters II and VII in terms of state sums on shadows. These results allow us to show that $|M| = \tau(M)\,\tau(-M)$. Sections X.4 - X.6 are devoted to the proof of a theorem used in Section X.3. Note the key role of Section X.5 where we compute the invariant F of links in $\mathbb{R}^3$ in terms of $6j$-symbols. In Sections X.7 and X.8 we relate the TQFT's constructed in Chapters IV and VII. Finally, in Section X.9 we use the technique of shadows to compute the invariant τ for graph 3-manifolds.

In Chapter XI we explain how quantum groups give rise to modular categories. We begin with a general discussion of quasitriangular Hopf algebras, ribbon Hopf algebras, and modular Hopf algebras (Sections XI.1 - XI.3 and XI.5). In order to derive modular categories from quantum groups we use more general quasimodular categories (Section XI.4). In Section XI.6 we outline relevant results from the theory of quantum groups at roots of unity and explain how to obtain modular categories. For completeness, we also discuss quantum groups with generic parameter; they give rise to semisimple ribbon categories (Section XI.7).

In Chapter XII we give a geometric construction of the modular categories determined by the quantum group $U_q(sl_2(\mathbb{C}))$ at roots of unity. The corner-stone of this approach is the skein theory of tangle diagrams (Sections XII.1 and XII.2) and a study of idempotents in the Temperley-Lieb algebras (Sections XII.3 and XII.4). After some preliminaries in Sections XII.5 and XII.6 we construct modular skein categories in Section XII.7. These categories are studied in the next two sections where we compute multiplicity modules and discuss when these categories are unitary.

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Part I

Towards Topological Field Theory

Chapter I
Invariants of graphs in Euclidean 3-space

1. Ribbon categories

1.0. Outline. We introduce ribbon categories forming the algebraic base of the theory presented in this book. These are monoidal categories (i.e., categories with tensor product) endowed with braiding, twist, and duality. All these notions are discussed here in detail; they will be used throughout the book. We also introduce an elementary graphical calculus allowing us to use drawings in order to present morphisms in ribbon categories.

As we shall see in Section 2, each ribbon category gives rise to a kind of "topological field theory" for links in Euclidean 3-space. In order to extend this theory to links in other 3-manifolds and to construct 3-dimensional TQFT's we shall eventually restrict ourselves to more subtle modular categories.

The definition of ribbon category has been, to a great extent, inspired by the theory of quantum groups. The reader acquainted with this theory may notice that braiding plays the role of the universal R-matrix of a quantum group (cf. Chapter XI).

1.1. Monoidal categories. The definition of a monoidal category axiomatizes the properties of the tensor product of modules over a commutative ring. Here we recall briefly the concepts of category and monoidal category, referring for details to [Ma2].

A category $\mathcal{V}$ consists of a class of objects, a class of morphisms, and a composition law for the morphisms which satisfy the following axioms. To each morphism f there are associated two objects of $\mathcal{V}$ denoted by source(f) and target(f). (One uses the notation f : source(f) $\to$ target(f).) For any objects V, W of $\mathcal{V}$, the morphisms $V \to W$ form a set denoted by $\mathrm{Hom}(V, W)$. The composition $f \circ g$ of two morphisms is defined whenever target(g) = source(f). This composition is a morphism source(g) $\to$ target(f). Composition is associative:

$$(f \circ g) \circ h = f \circ (g \circ h) \tag{1.1.a}$$

whenever both sides of this formula are defined. Finally, for each object V, there is a morphism $\mathrm{id}_V : V \to V$ such that

$$f \circ \mathrm{id}_V = f, \quad \mathrm{id}_V \circ g = g \tag{1.1.b}$$

for any morphisms $f : V \to W$, $g : W \to V$.

A tensor product in a category $\mathcal{V}$ is a covariant functor $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ which associates to each pair of objects V, W of $\mathcal{V}$ an object $V \otimes W$ of $\mathcal{V}$ and to each pair of morphisms $f : V \to V'$, $g : W \to W'$ a morphism $f \otimes g : V \otimes W \to V' \otimes W'$. To say that $\otimes$ is a covariant functor means that we have the following identities

(1.1.c) $$(f \circ f') \otimes (g \circ g') = (f \otimes g) \circ (f' \otimes g'),$$

(1.1.d) $$\mathrm{id}_V \otimes \mathrm{id}_W = \mathrm{id}_{V \otimes W}.$$

A strict monoidal category is a category $\mathcal{V}$ equipped with a tensor product and an object $\mathbb{1} = \mathbb{1}_{\mathcal{V}}$, called the unit object, such that the following conditions hold. For any object V of $\mathcal{V}$, we have

(1.1.e) $$V \otimes \mathbb{1} = V, \quad \mathbb{1} \otimes V = V$$

and for any triple U, V, W of objects of $\mathcal{V}$, we have

(1.1.f) $$(U \otimes V) \otimes W = U \otimes (V \otimes W).$$

For any morphism f in $\mathcal{V}$,

(1.1.g) $$f \otimes \mathrm{id}_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}} \otimes f = f$$

and for any triple f, g, h of morphisms in $\mathcal{V}$,

(1.1.h) $$(f \otimes g) \otimes h = f \otimes (g \otimes h).$$

More general (not necessarily strict) monoidal categories are defined similarly to strict monoidal categories though instead of (1.1.e), (1.1.f) one assumes that the right-hand sides and left-hand sides of these equalities are related by fixed isomorphisms. (A morphism $f : V \to W$ of a category is said to be an isomorphism if there exists a morphism $g : W \to V$ such that $fg = \mathrm{id}_W$ and $gf = \mathrm{id}_V$). These fixed isomorphisms should satisfy two compatibility conditions called the pentagon and triangle identities, see [Ma2]. These isomorphisms should also appear in (1.1.g) and (1.1.h) in the obvious way. For instance, the category of modules over a commutative ring with the standard tensor product of modules is monoidal. The ground ring regarded as a module over itself plays the role of the unit object. Note that this monoidal category is not strict. Indeed, if U, V, and W are modules over a commutative ring then the modules $(U \otimes V) \otimes W$ and $U \otimes (V \otimes W)$ are canonically isomorphic but not identical.

We shall be concerned mainly with strict monoidal categories. This does not lead to a loss of generality because of MacLane's coherence theorem which establishes equivalence of any monoidal category to a certain strict monoidal category. In particular, the category of modules over a commutative ring is equivalent to a strict monoidal category. Non-strict monoidal categories will essentially appear only in this section, in Section II.1, and in Chapter XI. Working with non-strict monoidal categories, we shall supress the fixed isomorphisms relating the right-hand sides and left-hand sides of equalities (1.1.e), (1.1.f). (Such abuse of notation is traditional in linear algebra.)

1.2. Braiding and twist in monoidal categories. The tensor multiplication of modules over a commutative ring is commutative in the sense that for any modules V, W, there is a canonical isomorphism $V \otimes W \to W \otimes V$. This isomorphism transforms any vector $v \otimes w$ into $w \otimes v$ and extends to $V \otimes W$ by linearity. It is called the flip and denoted by $P_{V,W}$. The system of flips is compatible with the tensor product in the obvious way: for any three modules U, V, W, we have

$$P_{U,V\otimes W} = (\mathrm{id}_V \otimes P_{U,W})(P_{U,V} \otimes \mathrm{id}_W), \quad P_{U\otimes V,W} = (P_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes P_{V,W}).$$

The system of flips is involutive in the sense that $P_{W,V}P_{V,W} = \mathrm{id}_{V\otimes W}$. Axiomatization of these properties of flips leads to the notions of a braiding and a twist in monoidal categories. From the topological point of view, braiding and twist (together with the duality discussed below) form a minimal set of elementary blocks necessary and sufficient to build up a topological field theory for links in $\mathbb{R}^3$.

A braiding in a monoidal category $\mathcal{V}$ consists of a natural family of isomorphisms

$$c = \{c_{V,W} : V \otimes W \to W \otimes V\}, \tag{1.2.a}$$

where V, W run over all objects of $\mathcal{V}$, such that for any three objects U, V, W, we have

$$c_{U,V\otimes W} = (\mathrm{id}_V \otimes c_{U,W})(c_{U,V} \otimes \mathrm{id}_W), \tag{1.2.b}$$

$$c_{U\otimes V,W} = (c_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes c_{V,W}). \tag{1.2.c}$$

(The reader is recommended to draw the corresponding commutative diagrams.) The naturality of the isomorphisms (1.2.a) means that for any morphisms $f : V \to V'$, $g : W \to W'$, we have

$$(g \otimes f)\, c_{V,W} = c_{V',W'}(f \otimes g). \tag{1.2.d}$$

Applying (1.2.b), (1.2.c) to $V = W = \mathbb{1}$ and $U = V = \mathbb{1}$ and using the invertibility of $c_{V,\mathbb{1}}, c_{\mathbb{1},V}$, we get

$$c_{V,\mathbb{1}} = c_{\mathbb{1},V} = \mathrm{id}_V \tag{1.2.e}$$

for any object V of $\mathcal{V}$. In Section 1.6 we shall show that any braiding satisfies the following Yang-Baxter identity:

$$\begin{aligned}(\mathrm{id}_W \otimes c_{U,V})\,(c_{U,W} \otimes \mathrm{id}_V)\,(\mathrm{id}_U \otimes c_{V,W}) &= \\ = (c_{V,W} \otimes \mathrm{id}_U)\,(\mathrm{id}_V \otimes c_{U,W})\,(c_{U,V} \otimes \mathrm{id}_W).\end{aligned} \tag{1.2.f}$$

Axiomatization of the involutivity of flips is slightly more involved. It would be too restrictive to require the composition $c_{W,V}c_{V,W}$ to be equal to $\mathrm{id}_{V\otimes W}$. What suits our aims better is to require this composition to be a kind of coboundary. This suggests the notion of a twist as follows. A twist in a monoidal category $\mathcal{V}$ with a braiding c consists of a natural family of isomorphisms

$$\theta = \{\theta_V : V \to V\}, \tag{1.2.g}$$

where V runs over all objects of $\mathcal{V}$, such that for any two objects V, W of $\mathcal{V}$, we have

$$\theta_{V\otimes W} = c_{W,V}\, c_{V,W}\, (\theta_V \otimes \theta_W). \tag{1.2.h}$$

The naturality of θ means that for any morphism $f : U \to V$ in $\mathcal{V}$, we have $\theta_V f = f\theta_U$. Using the naturality of the braiding, we may rewrite (1.2.h) as follows:

$$\theta_{V\otimes W} = c_{W,V}\, (\theta_W \otimes \theta_V)\, c_{V,W} = (\theta_V \otimes \theta_W)\, c_{W,V}\, c_{V,W}.$$

Note that $\theta_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}$. This follows from invertibility of $\theta_{\mathbb{1}}$ and the formula

$$(\theta_{\mathbb{1}})^2 = (\theta_{\mathbb{1}} \otimes \mathrm{id}_{\mathbb{1}})(\mathrm{id}_{\mathbb{1}} \otimes \theta_{\mathbb{1}}) = \theta_{\mathbb{1}} \otimes \theta_{\mathbb{1}} = \theta_{\mathbb{1}}.$$

These equalities follow respectively from (1.1.g), (1.1.c) and (1.1.b), (1.2.h) and (1.2.e) where we substitute $V = W = \mathbb{1}$.

1.3. Duality in monoidal categories. Duality in monoidal categories is meant to axiomatize duality for modules usually formulated in terms of non-degenerate bilinear forms. Of course, the general definition of duality should avoid the term "linear". It rather axiomatizes the properties of the evaluation pairing and co-pairing (cf. Lemma III.2.2).

Let $\mathcal{V}$ be a monoidal category. Assume that to each object V of $\mathcal{V}$ there are associated an object V^* of $\mathcal{V}$ and two morphisms

$$b_V : \mathbb{1} \to V \otimes V^*, \quad d_V : V^* \otimes V \to \mathbb{1}. \tag{1.3.a}$$

The rule $V \mapsto (V^*, b_V, d_V)$ is called a duality in $\mathcal{V}$ if the following identities are satisfied:

$$(\mathrm{id}_V \otimes d_V)(b_V \otimes \mathrm{id}_V) = \mathrm{id}_V, \tag{1.3.b}$$

$$(d_V \otimes \mathrm{id}_{V^*})(\mathrm{id}_{V^*} \otimes b_V) = \mathrm{id}_{V^*}. \tag{1.3.c}$$

Note that we do not require that $(V^*)^* = V$.

We need only one axiom relating the duality morphisms b_V, d_V with braiding and twist. We say that the duality in $\mathcal{V}$ is compatible with the braiding c and the twist θ in $\mathcal{V}$ if for any object V of $\mathcal{V}$, we have

$$(\theta_V \otimes \mathrm{id}_{V^*})\, b_V = (\mathrm{id}_V \otimes \theta_{V^*})\, b_V. \tag{1.3.d}$$

The compatibility leads to a number of implications pertaining to duality. In particular, we shall show in Section 2 that any duality in $\mathcal{V}$ compatible with braiding and twist is involutive in the sense that $V^{**} = (V^*)^*$ is canonically isomorphic to V.

1.4. Ribbon categories. By a ribbon category, we mean a monoidal category $\mathcal{V}$ equipped with a braiding c, a twist θ, and a compatible duality $(*, b, d)$. A ribbon category is called strict if its underlying monoidal category is strict.

Fundamental examples of ribbon categories are provided by the theory of quantum groups: Finite-dimensional representations of a quantum group form a ribbon category. For details, see Chapter XI.

To each ribbon category $\mathcal{V}$ we associate a mirror ribbon category $\overline{\mathcal{V}}$. It has the same underlying monoidal category and the same duality $(*, b, d)$. The braiding $\overline{c}$ and the twist $\overline{\theta}$ in $\overline{\mathcal{V}}$ are defined by the formulas

$$\overline{c}_{V,W} = (c_{W,V})^{-1} \text{ and } \overline{\theta}_V = (\theta_V)^{-1} \tag{1.4.a}$$

where c and θ are the braiding and the twist in $\mathcal{V}$. The axioms of ribbon category for $\overline{\mathcal{V}}$ follow directly from the corresponding axioms for $\mathcal{V}$.

MacLane's coherence theorem that establishes equivalence of any monoidal category to a strict monoidal category works in the setting of ribbon categories as well (cf. Remark XI.1.4). This enables us to focus attention on strict ribbon categories: all results obtained below for these categories directly extend to arbitrary ribbon categories.

1.5. Traces and dimensions. Ribbon categories admit a consistent theory of traces of morphisms and dimensions of objects. This is one of the most important features of ribbon categories sharply distinguishing them from arbitrary monoidal categories. We shall systematically use these traces and dimensions.

Let $\mathcal{V}$ be a ribbon category. Denote by $K = K_{\mathcal{V}}$ the semigroup $\mathrm{End}(\mathbb{1})$ with multiplication induced by the composition of morphisms and the unit element $\mathrm{id}_{\mathbb{1}}$. The semigroup K is commutative because for any morphisms $k, k' : \mathbb{1} \to \mathbb{1}$, we have

$$kk' = (k \otimes \mathrm{id}_{\mathbb{1}})(\mathrm{id}_{\mathbb{1}} \otimes k') = k \otimes k' = (\mathrm{id}_{\mathbb{1}} \otimes k')(k \otimes \mathrm{id}_{\mathbb{1}}) = k'k.$$

The traces of morphisms and the dimensions of objects which we define below take their values in K.

For an endomorphism $f : V \to V$ of an object V, we define its trace $\mathrm{tr}(f) \in K$ to be the following composition:

$$\mathrm{tr}(f) = d_V\, c_{V,V^*}((\theta_V f) \otimes \mathrm{id}_{V^*})\, b_V : \mathbb{1} \to \mathbb{1}. \tag{1.5.a}$$

For an object V of $\mathcal{V}$, we define its dimension $\dim(V)$ by the formula

$$\dim(V) = \mathrm{tr}(\mathrm{id}_V) = d_V\, c_{V,V^*}(\theta_V \otimes \mathrm{id}_{V^*})\, b_V \in K.$$

The main properties of the trace are collected in the following lemma which is proven in Section 2.

1.5.1. Lemma. (i) *For any morphisms* $f : V \to W$, $g : W \to V$, *we have* $\mathrm{tr}(fg) = \mathrm{tr}(gf)$.

(ii) *For any endomorphisms* f, g *of objects of* $\mathcal{V}$, *we have* $\mathrm{tr}(f \otimes g) = \mathrm{tr}(f)\mathrm{tr}(g)$.

(iii) *For any morphism* $k : \mathbb{1} \to \mathbb{1}$, *we have* $\mathrm{tr}(k) = k$.

The first claim of this lemma implies the naturality of the trace: for any isomorphism $g : W \to V$ and any $f \in \mathrm{End}(V)$,

$$\mathrm{tr}(g^{-1} f g) = \mathrm{tr}(f). \tag{1.5.b}$$

Lemma 1.5.1 implies fundamental properties of dim:

(i)′ isomorphic objects have equal dimensions,

(ii)′ for any objects V, W, we have $\dim(V \otimes W) = \dim(V)\dim(W)$, and

(iii)′ $\dim(\mathbb{1}) = 1$.

We shall show in Section 2 that $\dim(V^*) = \dim(V)$.

1.6. Graphical calculus for morphisms. Let $\mathcal{V}$ be a strict ribbon category. We describe a pictorial technique used to present morphisms in $\mathcal{V}$ by plane diagrams. This pictorial calculus will allow us to replace algebraic arguments involving commutative diagrams by simple geometric reasoning. This subsection serves as an elementary introduction to operator invariants of ribbon graphs introduced in Section 2.

A morphism $f : V \to W$ in the category $\mathcal{V}$ may be represented by a box with two vertical arrows oriented downwards, see Figure 1.1.

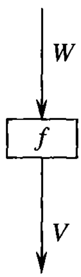


Figure 1.1

Here V, W should be regarded as "colors" of the arrows and f should be regarded as a color of the box. (Such boxes are called coupons.) More generally, a morphism $f : V_1 \otimes \ldots \otimes V_m \to W_1 \otimes \ldots \otimes W_n$ may be represented by a picture as in Figure 1.2. We do not exclude the case $m = 0$, or $n = 0$, or $m = n = 0$; by definition, for $m = 0$, the tensor product of m objects of $\mathcal{V}$ is equal to $\mathbb{1} = \mathbb{1}_{\mathcal{V}}$.

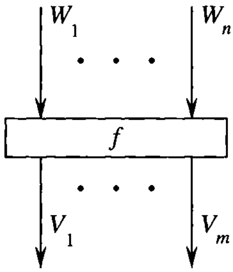


Figure 1.2

We shall use also vertical arrows oriented upwards under the convention that the morphism sitting in a box attached to such an arrow involves not the color of the arrow but rather the dual object. For example, a morphism $f : V^* \to W^*$ may be represented in four different ways, see Figure 1.3. From now on the symbol $\doteq$ displayed in the figures denotes equality of the corresponding morphisms in $\mathcal{V}$.

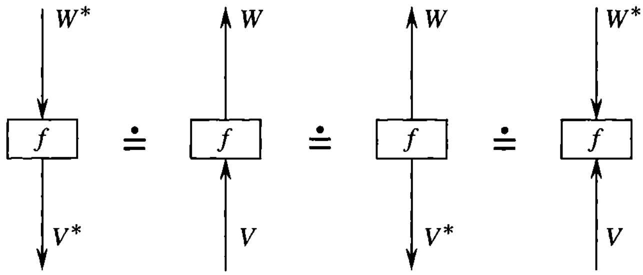


Figure 1.3

The identity endomorphism of any object V will be represented by a vertical arrow directed downwards and colored with V. A vertical arrow directed upwards and colored with V represents the identity endomorphism of V^*, see Figure 1.4.

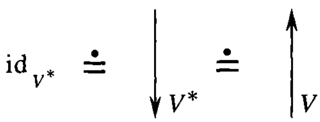


Figure 1.4

Note that a vertical arrow colored with $\mathbb{1}$ may be effaced from any picture without changing the morphism represented by this picture. We agree that the empty picture represents the identity endomorphism of $\mathbb{1}$.

The tensor product of two morphisms is presented as follows: just place a picture of the first morphism to the left of a picture of the second morphism. A picture for the composition of two morphisms f and g is obtained by putting a picture of f on the top of a picture of g and gluing the corresponding free ends of arrows. (Of course, this procedure may be applied only when the numbers of arrows, as well as their directions and colors are compatible.) In order to make this gluing smooth we should draw the arrows so that their ends are strictly vertical. For example, for any morphisms $f : V \to V'$ and $g : W \to W'$, the identities

$$(f \otimes \mathrm{id}_{W'})(\mathrm{id}_V \otimes g) = f \otimes g = (\mathrm{id}_{V'} \otimes g)(f \otimes \mathrm{id}_W)$$

have a graphical incarnation shown in Figure 1.5.

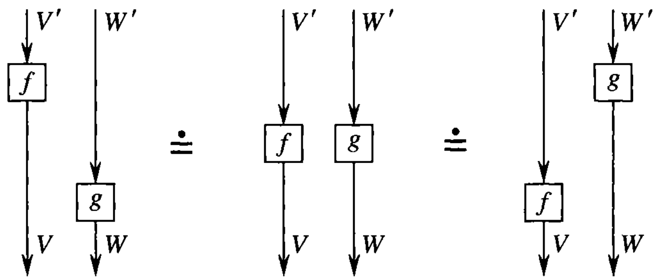


Figure 1.5

The braiding morphism $c_{V,W} : V \otimes W \to W \otimes V$ and the inverse morphism $c_{V,W}^{-1} : W \otimes V \to V \otimes W$ are represented by the pictures in Figure 1.6. Note that the colors of arrows do not change when arrows pass a crossing. The colors may change only when arrows hit coupons.

A graphical form of equalities (1.2.b), (1.2.c), (1.2.d) is given in Figure 1.7.

Using this notation, it is easy to verify the Yang-Baxter identity (1.2.f), see Figure 1.8 where we apply twice (1.2.b) and (1.2.d). Here is an algebraic form of the same argument:

$$(\mathrm{id}_W \otimes c_{U,V})(c_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes c_{V,W}) = c_{U,W\otimes V}(\mathrm{id}_U \otimes c_{V,W}) =$$

$$= (c_{V,W} \otimes \mathrm{id}_U)\, c_{U,V\otimes W} = (c_{V,W} \otimes \mathrm{id}_U)(\mathrm{id}_V \otimes c_{U,W})(c_{U,V} \otimes \mathrm{id}_W).$$

Using coupons colored with identity endomorphisms of objects, we may give different graphical forms to the same equality of morphisms in $\mathcal{V}$. In Figure 1.9 we give two graphical forms of (1.2.b). Here $\mathrm{id} = \mathrm{id}_{V\otimes W}$. For instance, the upper picture in Figure 1.9 presents the equality

$$c_{U,V\otimes W}(\mathrm{id}_U \otimes \mathrm{id}_{V\otimes W}) = (\mathrm{id}_{V\otimes W} \otimes \mathrm{id}_U)(\mathrm{id}_V \otimes c_{U,W})(c_{U,V} \otimes \mathrm{id}_W)$$

which is equivalent to (1.2.b). It is left to the reader to give similar reformulations of (1.2.c) and to draw the corresponding figures.

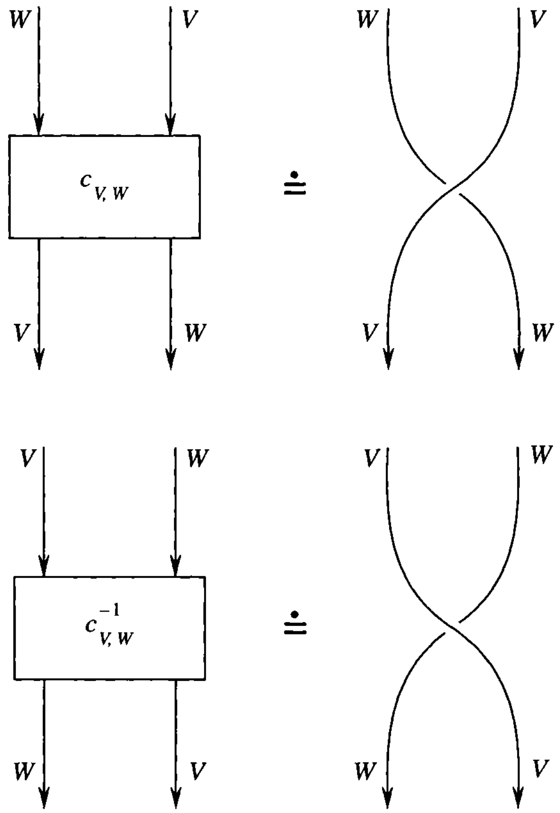


Figure 1.6

Duality morphisms $b_V : \mathbb{1} \to V \otimes V^*$ and $d_V : V^* \otimes V \to \mathbb{1}$ will be represented by the right-oriented cup and cap shown in Figure 1.10. For a graphical form of the identities (1.3.b), (1.3.c), see Figure 1.11.

The graphical technique outlined above applies to diagrams with only right-oriented cups and caps. In Section 2 we shall eliminate this constraint, describe a standard picture for the twist, and further generalize the technique. More importantly, we shall transform this pictorial calculus from a sort of skillful art into a concrete mathematical theorem.

1.7. Elementary examples of ribbon categories. We shall illustrate the concept of ribbon category with two simple examples. For more elaborate examples, see Chapters XI and XII.

1. Let K be a commutative ring with unit. By a projective K-module, we mean a finitely generated projective K-module, i.e., a direct summand of K^n with finite $n = 0, 1, 2, \ldots$ For example, free K-modules of finite rank are projective. It is obvious that the tensor product of a finite number of projective modules is

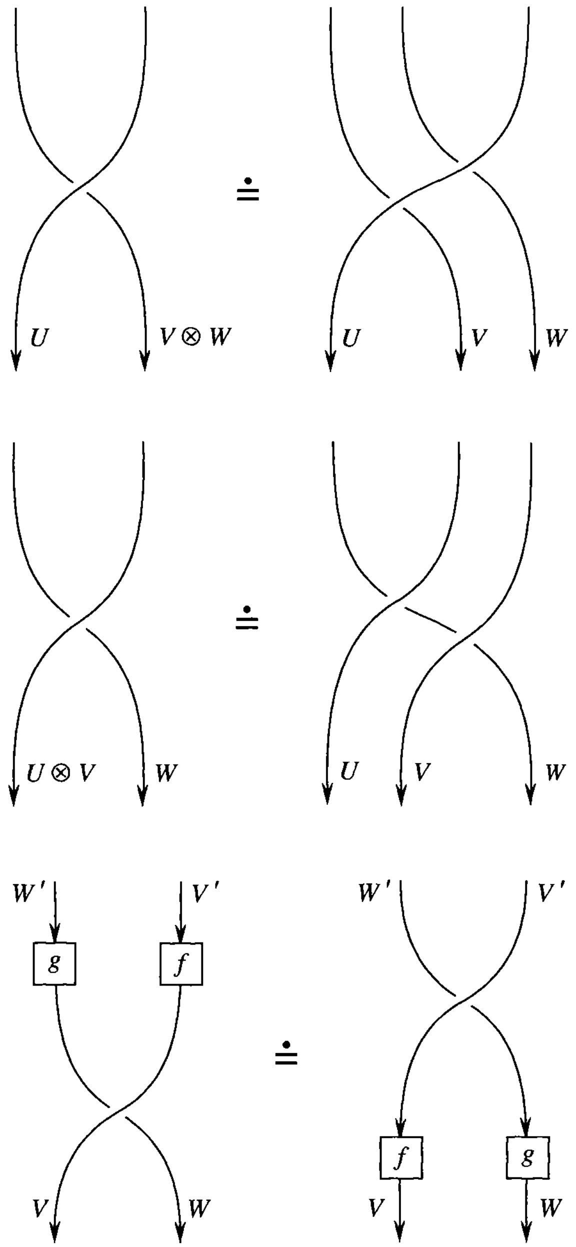


Figure 1.7

projective. For any projective K-module V, the dual K-module $V^\star = \mathrm{Hom}_K(V, K)$ is also projective and the canonical homomorphism $V \to V^{\star\star}$ is an isomorphism.

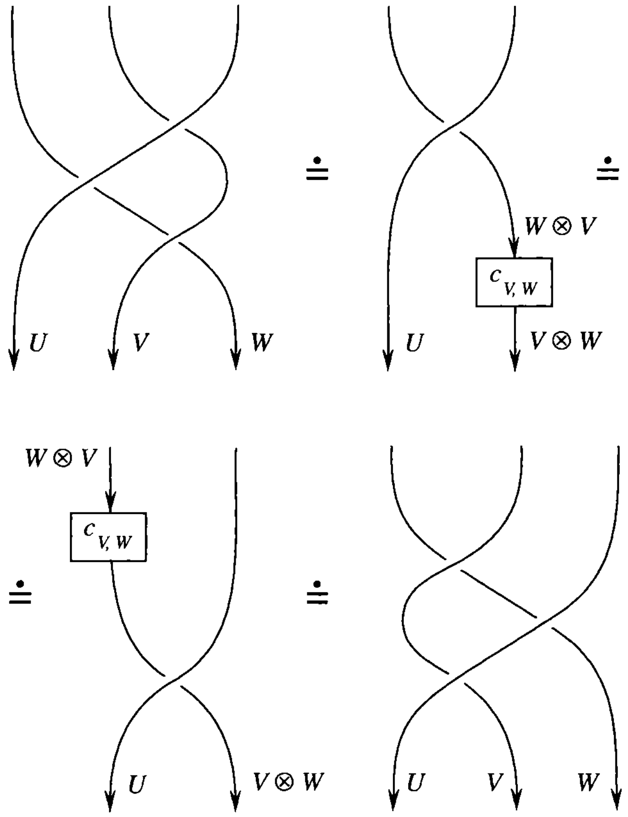


Figure 1.8

Let $\mathrm{Proj}(K)$ be the category of projective K-modules and K-linear homomorphisms. Provide $\mathrm{Proj}(K)$ with the usual tensor product over K. Set $\mathbb{1} = K$. It is obvious that $\mathrm{Proj}(K)$ is a monoidal category. We provide this category with braiding, twist, and duality. The braiding in $\mathrm{Proj}(K)$ is given by flips described in Section 1.2. The twist is given by the identity endomorphisms of objects. For any projective K-module V, set $V^* = V^\star = \mathrm{Hom}_K(V, K)$ and define d_V to be the evaluation pairing $v \otimes w \mapsto v(w) : V^\star \otimes V \to K$. Finally, define b_V to be the homomorphism $K \to V \otimes V^\star$ dual to $d_V : V^\star \otimes V \to K$ where we use the standard identifications $K^\star = K$ and $(V^\star \otimes V)^\star = V^{\star\star} \otimes V^\star = V \otimes V^\star$. The last two equalities follow from projectivity of V. (If V is a free module with a basis $\{e_i\}_i$ and $\{e^i\}_i$ is the dual basis of $V^\star$ then $b_V(1) = \sum_i e_i \otimes e^i$.) All axioms of ribbon categories are easily seen to be satisfied. Verification of (1.3.b) and (1.3.c) is an exercise in linear algebra, it is left to the reader.

The ribbon category $\mathrm{Proj}(K)$ is not interesting from the viewpoint of applications to knots. Indeed, we have $c_{V,W} = (c_{W,V})^{-1}$ so that the morphisms associ-

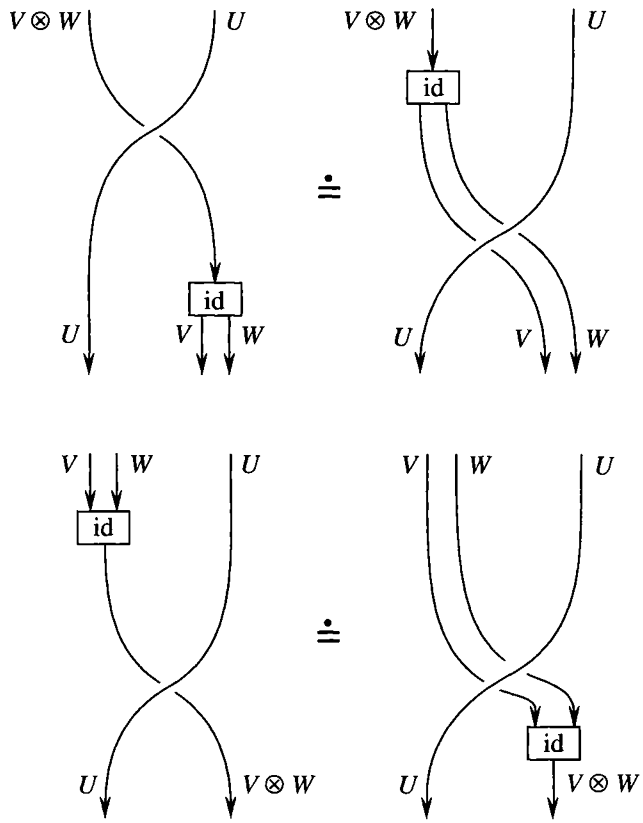


Figure 1.9

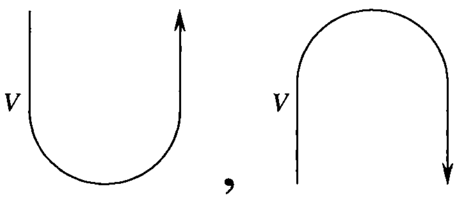


Figure 1.10

ated with diagrams are preserved under trading overcrossings for undercrossings, which kills the 3-dimensional topology of diagrams (cf. Figure 1.6).

Applying the definitions of Section 1.5 to the morphisms and objects of $\mathrm{Proj}(K)$ we get the notions of a dimension for projective K-modules and a trace for K-endomorphisms of projective K-modules. We shall denote these dimension and trace by Dim and Tr respectively. They generalize the usual dimension and trace for free modules and their endomorphisms (cf. Lemma II.4.3.1).

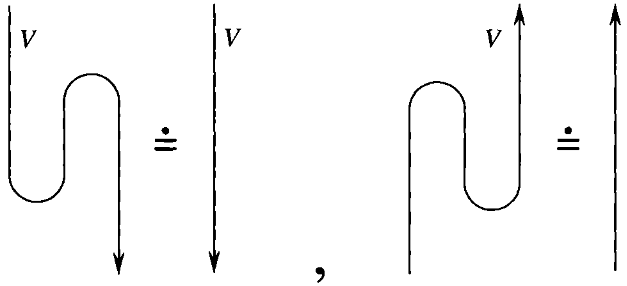


Figure 1.11

2. Let G be a multiplicative abelian group, K a commutative ring with unit, c a bilinear pairing $G \times G \to K^*$ where K^* is the multiplicative group of invertible elements of K. Thus $c(gg', h) = c(g, h)\, c(g', h)$ and $c(g, hh') = c(g, h)\, c(g, h')$ for any $g, g', h, h' \in G$. Using these data, we construct a ribbon category $\mathcal{V}$. The objects of $\mathcal{V}$ are elements of G. For any $g \in G$, the set of morphisms $g \to g$ is a copy of K. For distinct $g, h \in G$ the set of morphisms $g \to h$ consists of one element called zero. The composition of two morphisms $g \to h \to f$ is the product of the corresponding elements of K if $g = h = f$ and zero otherwise. The unit of K plays the role of the identity endomorphism of any object. The tensor product of $g, h \in G$ is defined to be their product $gh \in G$. The tensor product $gg' \to hh'$ of two morphisms $g \to h$ and $g' \to h'$ is the product of the corresponding elements of K if $g = h$ and $g' = h'$ and zero otherwise. It is easy to check that $\mathcal{V}$ is a strict monoidal category with the unit object being the unit of G. For $g, h \in G$, we define the braiding $gh \to hg = gh$ to be $c(g, h) \in K$ and the twist $g \to g$ to be $c(g, g) \in K$. Equalities (1.2.b), (1.2.c), and (1.2.h) follow from bilinearity of c. The naturality of the braiding and twist is straightforward. For $g \in G$, the dual object g^* is defined to be the inverse $g^{-1} \in G$ of g. Morphisms (1.3.a) are endomorphisms of the unit of G represented by $1 \in K$. Equalities (1.3.b) and (1.3.c) are straightforward. Formula (1.3.d) follows from the identity $c(g^{-1}, g^{-1}) = c(g, g)$. Thus, $\mathcal{V}$ is a ribbon category.

We may slightly generalize the construction of $\mathcal{V}$. Besides G, K, c, fix a group homomorphism $\varphi : G \to K^*$ such that $\varphi(g^2) = 1$ for all $g \in G$. We define the braiding and duality as above but define the twist $g \to g$ to be $\varphi(g)\, c(g, g) \in K$. It is easy to check that this yields a ribbon category. (The assumption $\varphi(g^2) = 1$ ensures (1.3.d).) This ribbon category is denoted by $\mathcal{V}(G, K, c, \varphi)$. The case considered above corresponds to $\varphi = 1$.

1.8. Exercises. 1. Use the graphical calculus to show that for any three objects U, V, W of a ribbon category, the homomorphisms

$$f \mapsto (d_V \otimes \mathrm{id}_W)(\mathrm{id}_{V^*} \otimes f) : \mathrm{Hom}(U, V \otimes W) \to \mathrm{Hom}(V^* \otimes U, W)$$

and

$$g \mapsto (\mathrm{id}_V \otimes g)(b_V \otimes \mathrm{id}_U) : \mathrm{Hom}(V^* \otimes U, W) \to \mathrm{Hom}(U, V \otimes W)$$

are mutually inverse. This establishes a bijective correspondence between the sets $\mathrm{Hom}(U, V \otimes W)$ and $\mathrm{Hom}(V^* \otimes U, W)$. Write down similar formulas for a bijective correspondence between $\mathrm{Hom}(U \otimes V, W)$ and $\mathrm{Hom}(U, W \otimes V^*)$.

2. Define the dual $f^* : V^* \to U^*$ of a morphism $f : U \to V$ by the formula

$$f^* = (d_V \otimes \mathrm{id}_{U^*})(\mathrm{id}_{V^*} \otimes f \otimes \mathrm{id}_{U^*})(\mathrm{id}_{V^*} \otimes b_U).$$

Give a pictorial interpretation of this formula. Use it to show that $(\mathrm{id}_V)^* = \mathrm{id}_{V^*}$ and $(fg)^* = g^* f^*$ for composable morphisms f, g. Show that (1.3.d) is equivalent to the formula

$$\theta_{V^*} = (\theta_V)^*. \tag{1.8.a}$$

3. Show that every duality in a monoidal category $\mathcal{V}$ is compatible with the tensor product in the sense that for any objects V, W of $\mathcal{V}$, the object $(V \otimes W)^*$ is isomorphic to $W^* \otimes V^*$. Set $U = V \otimes W$. Use the graphical calculus to show that the following morphisms are mutually inverse isomorphisms:

$$(d_U \otimes \mathrm{id}_{W^*} \otimes \mathrm{id}_{V^*})(\mathrm{id}_{U^*} \otimes \mathrm{id}_V \otimes b_W \otimes \mathrm{id}_{V^*})(\mathrm{id}_{U^*} \otimes b_V) : U^* \to W^* \otimes V^*,$$

$$(d_W \otimes \mathrm{id}_{U^*})(\mathrm{id}_{W^*} \otimes d_V \otimes \mathrm{id}_W \otimes \mathrm{id}_{U^*})(\mathrm{id}_{W^*} \otimes \mathrm{id}_{V^*} \otimes b_U) : W^* \otimes V^* \to U^*.$$

(Hint: use coupons colored with id_U.) Show that modulo these isomorphisms we have $(f \otimes g)^* = g^* \otimes f^*$ for any morphisms f, g in $\mathcal{V}$.

4. Use the graphical calculus to show that if $f : U \to V$ and $g : V \to U$ are mutually inverse morphisms in a ribbon category then $(f \otimes g^*)\, b_U = b_V$ and $d_V(g^* \otimes f) = d_U$.

2. Operator invariants of ribbon graphs

2.0. Outline. The objective of this section is to relate the theory of ribbon categories to the theory of links in Euclidean space $\mathbb{R}^3$. For technical reasons, it is convenient to deal with the strip $\mathbb{R}^2 \times [0, 1]$ rather than with $\mathbb{R}^3$. This does not lead to a loss of generality because any link in $\mathbb{R}^3$ may be deformed into $\mathbb{R}^2 \times [0, 1]$.

In generalization of links and braids we shall consider graphs embedded in $\mathbb{R}^2 \times [0, 1]$. In fact, instead of usual graphs formed by vertices and edges we shall consider ribbon graphs formed by small rectangles (coupons) and long bands. It is understood that the bands are attached to the bases of coupons and, possibly, to certain intervals in the planes $\mathbb{R}^2 \times 0$ and $\mathbb{R}^2 \times 1$. The bands attached to the last intervals are called free ends of the graph.

The next step is to marry the topology of ribbon graphs with the algebra of ribbon categories. To this end we introduce colorings of ribbon graphs by objects and morphisms of a given ribbon category $\mathcal{V}$. The bands are colored with objects whilst the coupons are colored with morphisms. The ribbon graphs with such colorings (or rather their isotopy classes) form a monoidal category $\mathrm{Rib}_{\mathcal{V}}$. The definition and study of the category of ribbon graphs marks one of the major steps towards the 3-dimensional topological field theory.

The main result of this section (Theorem 2.5) establishes the existence of a certain covariant functor $F : \mathrm{Rib}_{\mathcal{V}} \to \mathcal{V}$. The functor $F = F_{\mathcal{V}}$ should be viewed as a "topological field theory" in Euclidean 3-space. This functor will play a fundamental role in the book. It is instrumental in the construction of 3-manifold invariants in Chapter II, in the construction of 3-dimensional TQFT's in Chapter IV, as well as in the definition of normalized $6j$-symbols in Chapter VI. The functor F also provides a solid grounding for the pictorial calculus of Section 1.6.

Note that ribbon graphs (and not just links) are really important for the constructions in Chapters IV - VI. We demonstrate utility of coupons at the end of this section where we discuss a few simple properties and applications of F.

The definition of ribbon graphs and related notions are somewhat technical. They involve a number of small arrangements concerned with orientations, bases of coupons, free ends of graphs, etc. The reader should not focus all his attention on these details, it is more important to catch the general idea rather than technicalities.

2.1. Ribbon graphs and their diagrams. Roughly speaking, ribbon graphs are oriented compact surfaces in $\mathbb{R}^3$ decomposed into elementary pieces: bands, annuli, and coupons. We start with the formal definition of these pieces.

A band is the square $[0, 1] \times [0, 1]$ or a homeomorphic image of this square. The images of the intervals $[0, 1] \times 0$ and $[0, 1] \times 1$ are called bases of the band. The image of the interval $(1/2) \times [0, 1]$ is called the core of the band. An annulus is the cylinder $S^1 \times [0, 1]$ or a homeomorphic image of this cylinder. The image of the circle $S^1 \times (1/2)$ is called the core of the annulus. A band or an annulus is said to be directed if its core is oriented. The orientation of the core is called the direction of the band (resp. annuli). A coupon is a band with a distinguished base. This distinguished base is called the bottom base of the coupon, the opposite base is said to be the top one.

Let k, l be non-negative integers. We define ribbon graphs with k inputs and l outputs or, briefly, ribbon (k, l)-graphs. A ribbon (k, l)-graph in $\mathbb{R}^3$ is an oriented surface Ω embedded in the strip $\mathbb{R}^2 \times [0, 1]$ and decomposed into a union of a finite number of annuli, bands, and coupons such that

(i) Ω meets the planes $\mathbb{R}^2 \times 0$, $\mathbb{R}^2 \times 1$ orthogonally along the following segments which are bases of certain bands of Ω:

(2.1.a) $$\{ [i-(1/10), i+(1/10)] \times 0 \times 0 \mid i = 1, ..., k \},$$

(2.1.b) $$\{ [j-(1/10), j+(1/10)] \times 0 \times 1 \mid j = 1, ..., l \}.$$

In the points of these segments the orientation of Ω is determined by the pair of vectors $(1,0,0)$, $(0,0,1)$ tangent to Ω;

(ii) other bases of bands lie on the bases of coupons; otherwise the bands, coupons, and annuli are disjoint;

(iii) the bands and annuli of Ω are directed.

The surface Ω with the splitting into annuli, bands, and coupons forgotten is called the surface of the ribbon (k,l)-graph Ω. The intervals (2.1.a) (resp. (2.1.b)) are called bottom (resp. top) boundary intervals of the graph.

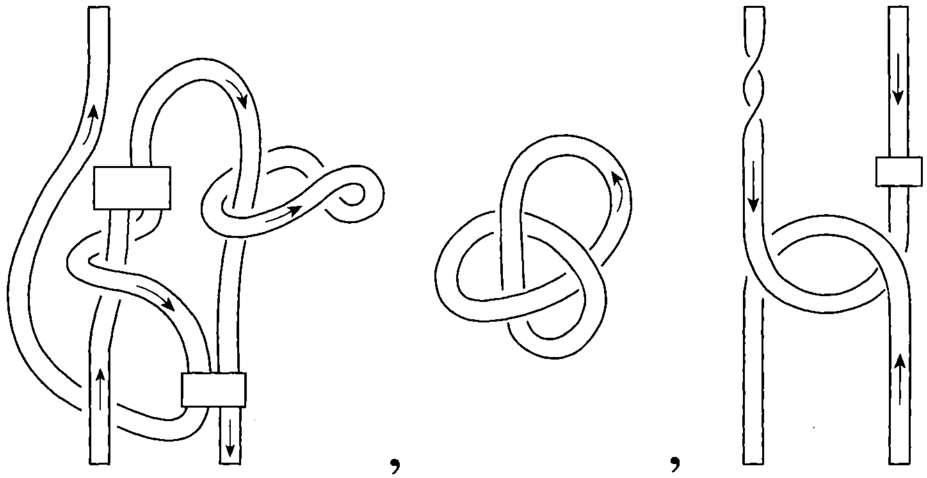
Each band should be thought of as a narrow strip or ribbon with short bases. The coupons lie in $\mathbb{R}^2 \times (0,1)$, each coupon should be thought of as a small rectangle with a distinguished base. Note that we impose no conditions on the geometric position of coupons in $\mathbb{R}^2 \times (0,1)$. In particular, the distinguished (bottom) bases of coupons may actually lie higher than the opposite bases. (Since we shall consider ribbon graphs up to isotopy we shall be able to avoid this in our pictures.)

The choice of orientation for the surface Ω of a ribbon graph is equivalent to a choice of a preferred side of Ω. (We fix the right-handed orientation in $\mathbb{R}^3$.) The orientation condition in (i) means that near the boundary intervals the preferred side of Ω is the one turned up, i.e., towards the reader.

By a ribbon graph, we mean a ribbon (k,l)-graph with $k, l \geq 0$. Examples of ribbon graphs are given in Figure 2.1 where the bottom bases of coupons are their lower horizontal bases and the preferred side of Ω is the one turned up.

By isotopy of ribbon graphs, we mean isotopy in the strip $\mathbb{R}^2 \times [0,1]$ constant on the boundary intervals and preserving the splitting into annuli, bands, and coupons, as well as preserving the directions of bands and annuli, and the orientation of the graph surface. Note that in the course of isotopy the bases of bands lying on the bases of coupons may move along these bases (not touching each other) but can not slide to the sides of coupons. Note also that when we rotate an annulus in $\mathbb{R}^3$ around its core by the angle of π we get the same annulus with the opposite orientation. Therefore, orientations of annuli are immaterial when we consider ribbon graphs up to isotopy.

There is a convenient technique enabling us to present ribbon graphs by plane pictures generalizing the standard knot diagrams. The idea is to deform the graph in $\mathbb{R}^2 \times [0,1]$ into a "standard position" so that it lies almost parallel and very close to the plane $\mathbb{R} \times 0 \times \mathbb{R}$ as in Figures 2.1.a and 2.1.b. (The plane $\mathbb{R} \times 0 \times \mathbb{R}$ is identified with the plane of the pictures.) In particular, the coupons should be



(a) (b) (c)

Figure 2.1

plane rectangles parallel to $\mathbb{R} \times 0 \times \mathbb{R}$. The bases of coupons should be parallel to the horizontal line $\mathbb{R} \times 0 \times 0$ and the top base of each coupon should lie higher than the bottom one. The orientation of coupons induced by the orientation of Ω should correspond to the counterclockwise orientation in $\mathbb{R} \times 0 \times \mathbb{R}$ (so that the preferred side of each coupon is turned towards the reader). The bands and annuli of the graph should go close and "parallel" to this plane. The projections of the cores of bands and annuli in the plane $\mathbb{R} \times 0 \times \mathbb{R}$ should have only double transversal crossings and should not overlap with the projections of coupons. After having deformed the graph in such a position we draw the projections of the coupons and the cores of the bands and annuli in $\mathbb{R} \times 0 \times \mathbb{R}$ taking into account the overcrossings and undercrossings of the cores. The projections of the cores of bands and annuli are oriented in accordance with their directions. The resulting picture is called a diagram of the ribbon graph.

Looking at such a diagram we may reconstruct the original ribbon graph (up to isotopy) just by letting the bands and annuli go "parallel" to the plane of the picture along their cores. One may think that arcs in our diagrams have some small width so that actually we draw very thin bands and annuli. For example, the graph diagrams in Figure 2.2 present the same ribbon graphs as in Figure 2.1, (a) and (b).

The technique of graph diagrams is sufficiently general: any ribbon graph is isotopic to a ribbon graph lying in a standard position (as described above) and therefore presented by a graph diagram. To see this, we first deform the graph so that its coupons lie in a standard position and then we deform the bands so that they go "parallel" to the plane of the picture. The only problem which we

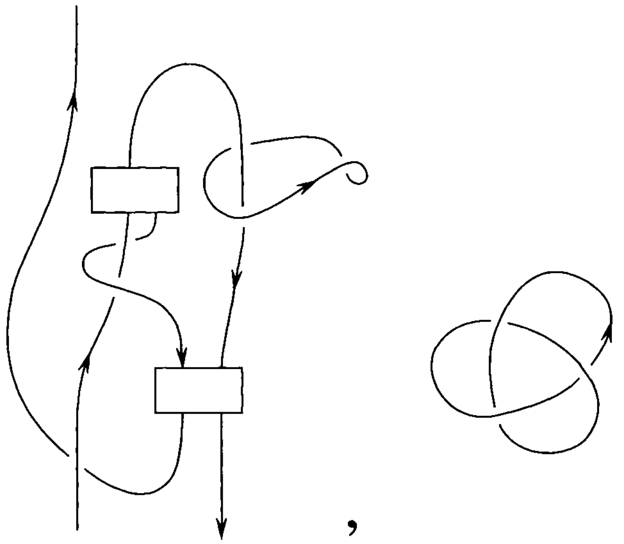


Figure 2.2

may encounter is that the bands may be twisted several times around their cores. However, both positive and negative twists in a band are isotopic to cirls which go "parallel" to the plane. See Figure 2.3 which presents positive and negative twists in a band. (The symbol $\approx$ denotes isotopy.) Note that positivity of the twist does not depend on the direction of the band and depends solely on the orientation of the ambient 3-manifold; we use everywhere the right-handed orientation in $\mathbb{R}^3$. Annuli are treated in a similar way.

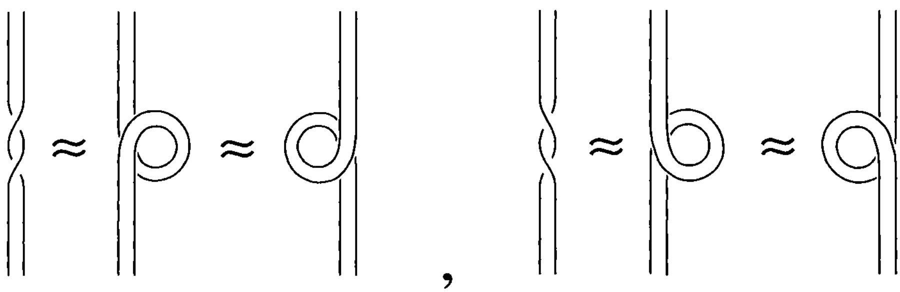


Figure 2.3

The theory of ribbon graphs generalizes the more familiar theory of framed oriented links in $\mathbb{R}^3$. A link L in $\mathbb{R}^3$ is a finite collection of smooth disjoint embedded circles $L_1, \ldots, L_m \subset \mathbb{R}^3$. The link L is oriented if its components $L_1, \ldots, L_m$ are oriented. The link L is framed if it is endowed with a homotopy class of non-singular normal vector fields on $L_1, ..., L_m$ in $\mathbb{R}^3$. Note that the homotopy class of a non-singular normal vector field on a component L_i is completely determined by the rotation number of the field around L_i. (This is an integer defined as the linking number of L_i with the longitude $L_i' \subset \mathbb{R}^3 \setminus L_i$ obtained by pushing L_i along the normal vector field on L_i; to compute this linking number we need to orient L_i and provide L_i' with the induced orientation; the resulting linking number does not depend on the choice of orientation in L_i.) Therefore, in order to specify a framing on a link it suffices to assign an integer to each component. These integers are called framing numbers or framings.

To each ribbon (0,0)-graph Ω consisting of annuli we may associate the link of circles in $\mathbb{R}^3$ formed by the oriented cores of the annuli. These circles are provided with a normal vector field transversal to Ω. The resulting framing is correctly defined since different choices of the normal vector field lead to homotopic vector fields on the link. In this way we get a bijective correspondence between isotopy classes of ribbon (0,0)-graphs consisting of annuli and isotopy classes of framed oriented links in $\mathbb{R}^2 \times (0, 1)$. For instance, the ribbon graph drawn in Figure 2.1.b corresponds to the trefoil knot with the framing number -3.

2.2. Ribbon graphs over $\mathcal{V}$. Fix a strict monoidal category with duality $\mathcal{V}$. A ribbon graph is said to be colored (over $\mathcal{V}$) if each band and each annulus of the graph is equipped with an object of $\mathcal{V}$. This object is called the color of the band (the annulus).

The coupons of a ribbon graph may be colored by morphisms in $\mathcal{V}$. Let Q be a coupon of a colored ribbon graph Ω. Let $V_1, ..., V_m$ be the colors of the bands of Ω incident to the bottom base of Q and encountered in the order induced by the orientation of Ω restricted to Q (see Figure 2.4 where Q is oriented counterclockwise). Let $W_1, ..., W_n$ be the colors of the bands of Ω incident to the top base of Q and encountered in the order induced by the opposite orientation of Q. Let $\varepsilon_1, ..., \varepsilon_m \in \{+1, -1\}$ (resp. $\nu_1, ..., \nu_n \in \{1, -1\}$) be the numbers determined by the directions of these bands: $\varepsilon_i = +1$ (resp. $\nu_j = -1$) if the band is directed "out" of the coupon and $\varepsilon_i = -1$ (resp. $\nu_j = +1$) in the opposite case. A color of the coupon Q is an arbitrary morphism

$$f : V_1^{\varepsilon_1} \otimes ... \otimes V_m^{\varepsilon_m} \to W_1^{\nu_1} \otimes ... \otimes W_n^{\nu_n}$$

where for an object V of $\mathcal{V}$, we set $V^{+1} = V$ and $V^{-1} = V^*$. A ribbon graph is v-colored (over $\mathcal{V}$) if it is colored and all its coupons are provided with colors as above. It is in the definition of colorings of coupons that we need to distinguish bottom and top bases of coupons.

For example, Figure 2.4 presents a v-colored ribbon (m, n)-graph containing one coupon, $m + n$ vertical untwisted unlinked bands incident to this coupon, and no annuli. As above the signs $\varepsilon_1, ..., \varepsilon_m, \nu_1, ..., \nu_n \in \{+1, -1\}$ determine the directions of the bands (the band is directed downwards if the corresponding sign is $+1$ and upwards if the sign is -1). We shall call this ribbon graph an elementary

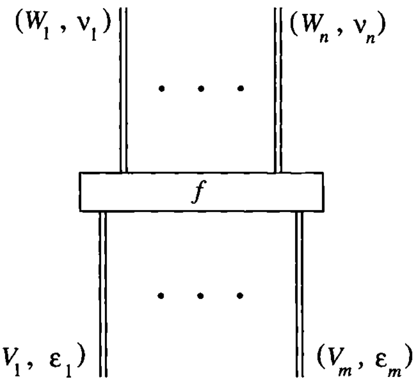


Figure 2.4

v-colored ribbon graph. Figure 2.1 gives a few examples of non-elementary ribbon graphs.

By isotopy of colored (resp. v-colored) ribbon graphs, we mean color-preserving isotopy.

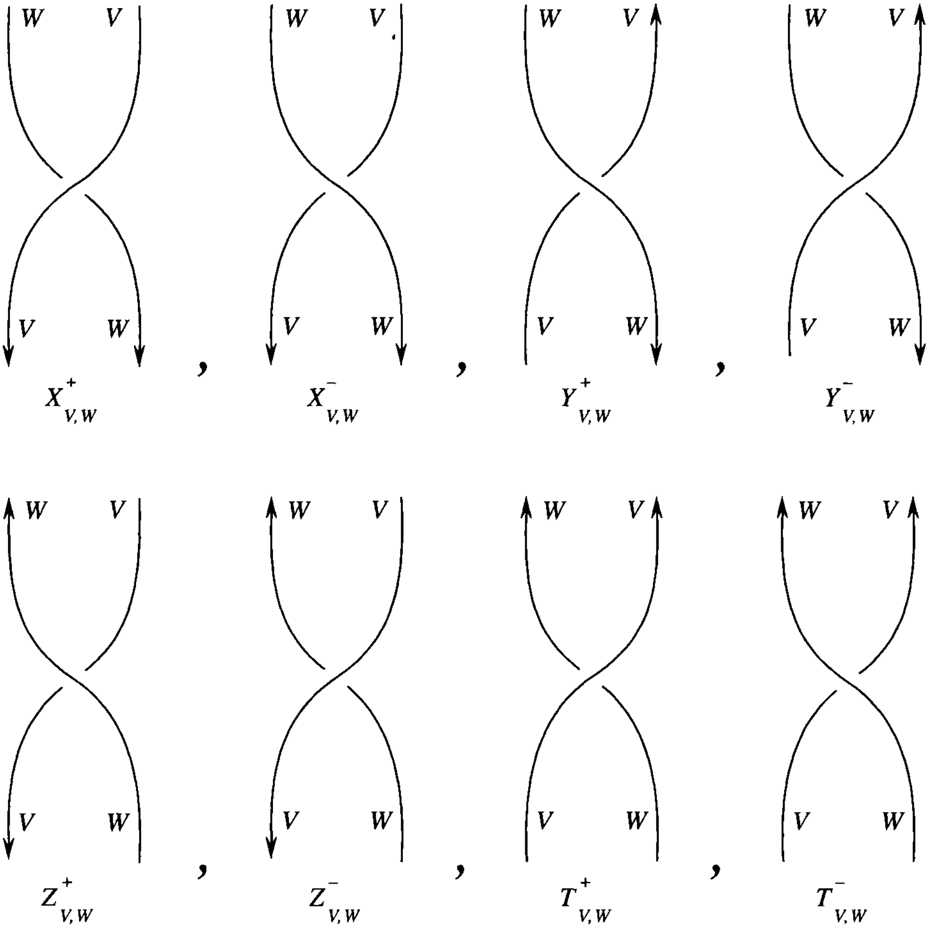
The technique of diagrams readily extends to colored and v-colored ribbon graphs. To present a colored ribbon graph by a diagram, we attach an object of $\mathcal{V}$ to the cores of bands and annuli. To present a v-colored ribbon graph, we additionally assign colors to all coupons.

The notions of colored and v-colored ribbon graphs at first glance seem to be artificial and eclectic. These notions mix topological and algebraic concepts in a seemingly arbitrary way. In particular, links may be colored in many different ways, leading to numerous link invariants (constructed below). However, it is precisely in this mix of topology and algebra that lies the novelty and strength of the theory. As a specific justification of this approach, note that the invariants of a framed link $L \subset \mathbb{R}^3$ corresponding to essentially all colorings of L may be combined to produce a single invariant of the 3-manifold obtained by surgery along L (see Chapter II).

2.3. Category of ribbon graphs over $\mathcal{V}$. Let $\mathcal{V}$ be a strict monoidal category with duality. The v-colored ribbon graphs over $\mathcal{V}$ may be organized into a strict monoidal category denoted by $\mathrm{Rib}_{\mathcal{V}}$. The objects of $\mathrm{Rib}_{\mathcal{V}}$ are finite sequences $((V_1, \varepsilon_1), ..., (V_m, \varepsilon_m))$ where $V_1, ..., V_m$ are objects of $\mathcal{V}$ and $\varepsilon_1, ..., \varepsilon_m \in \{+1, -1\}$. The empty sequence is also considered as an object of $\mathrm{Rib}_{\mathcal{V}}$. A morphism $\eta \to \eta'$ in $\mathrm{Rib}_{\mathcal{V}}$ is an isotopy type of a v-colored ribbon graph (over $\mathcal{V}$) such that η (resp. η') is the sequence of colors and directions of those bands which hit the bottom (resp. top) boundary intervals. As usual, $\varepsilon = 1$ corresponds to the downward direction near the corresponding boundary interval and $\varepsilon = -1$ corresponds to the band directed up. For example, the ribbon graph drawn in Figure 2.4 represents a morphism $((V_1, \varepsilon_1), ..., (V_m, \varepsilon_m)) \to ((W_1, \nu_1), ..., (W_n, \nu_n))$. It should be emphasized that isotopic v-colored ribbon graphs present the same morphism in $\mathrm{Rib}_{\mathcal{V}}$.

The composition of two morphisms $f : \eta \to \eta'$ and $g : \eta' \to \eta''$ is obtained by putting a v-colored ribbon graph representing g on the top of a ribbon graph representing f, gluing the corresponding ends, and compressing the result into $\mathbb{R}^2 \times [0, 1]$. The identity morphisms are represented by ribbon graphs which have no annuli and no coupons, and consist of untwisted unlinked vertical bands. The identity endomorphism of the empty sequence is represented by the empty ribbon graph.

We provide $\mathrm{Rib}_{\mathcal{V}}$ with a tensor multiplication. The tensor product of objects η and η' is their juxtaposition η, η'. The tensor product of morphisms f, g is ob-



$X^+_{V,W}$, $X^-_{V,W}$, $Y^+_{V,W}$, $Y^-_{V,W}$

$Z^+_{V,W}$, $Z^-_{V,W}$, $T^+_{V,W}$, $T^-_{V,W}$

Figure 2.5

tained by placing a v-colored ribbon graph representing f to the left of a v-colored ribbon graph representing g so that there is no mutual linking or intersection. It is obvious that this tensor multiplication makes $\mathrm{Rib}_{\mathcal{V}}$ a strict monoidal category.

We shall need certain specific morphisms in $\mathrm{Rib}_{\mathcal{V}}$ presented by graph diagrams in Figure 2.5 where we also specify notation for these morphisms. Here the colors of the strings V, W run over objects of $\mathcal{V}$. The morphisms in $\mathrm{Rib}_{\mathcal{V}}$ presented by the diagrams in Figure 2.6 will be denoted by $\downarrow_V, \uparrow_V, \varphi_V, \varphi'_V, \cap_V, \cap^-_V, \cup_V, \cup^-_V$, respectively.

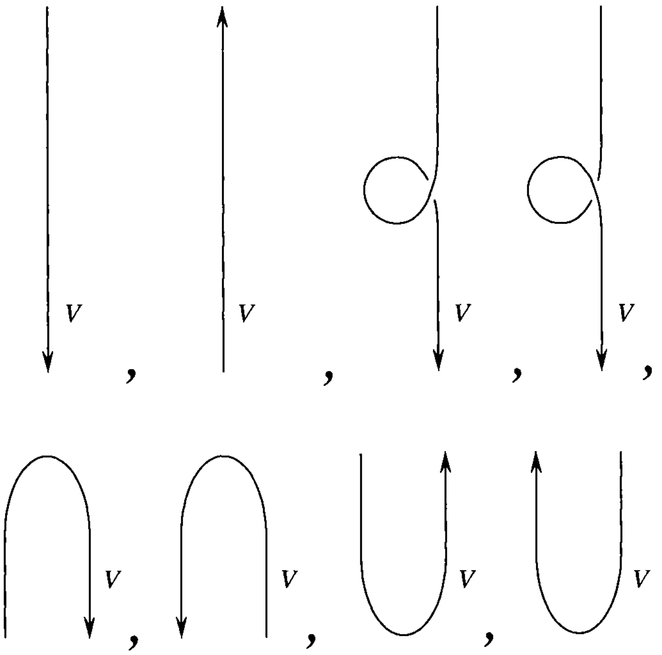


Figure 2.6

A ribbon graph over $\mathcal{V}$ which has no coupons is called a ribbon tangle over $\mathcal{V}$. It is obvious that ribbon tangles form a subcategory of $\mathrm{Rib}_{\mathcal{V}}$ which has the same objects as $\mathrm{Rib}_{\mathcal{V}}$ but less morphisms. This subcategory is called the category of colored ribbon tangles. It is a strict monoidal category under the same tensor product.

As the reader may have guessed, the category $\mathrm{Rib}_{\mathcal{V}}$ admits a natural braiding, twist, and duality and becomes in this way a ribbon category. We shall not use these structures in $\mathrm{Rib}_{\mathcal{V}}$ and do not discuss them. (For similar structures in a related setting, see Chapter XII.)

2.4. Digression on covariant functors. A covariant functor F of a category $\mathcal{X}$ into a category $\mathcal{Y}$ assigns to each object V of $\mathcal{X}$ an object $F(V)$ of $\mathcal{Y}$ and to each morphism $f : V \to W$ in $\mathscr{X}$ a morphism $F(f) : F(V) \to F(W)$ in $\mathscr{Y}$ so that $F(\mathrm{id}_V) = \mathrm{id}_{F(V)}$ for any object V of $\mathscr{X}$ and $F(fg) = F(f)F(g)$ for any two composable morphisms f, g in $\mathscr{X}$. If $\mathscr{X}$ and $\mathscr{Y}$ are monoidal categories then the covariant functor $F : \mathscr{X} \to Y$ is said to preserve the tensor product if $F(\mathbb{1}_{\mathscr{X}}) = \mathbb{1}_{\mathscr{Y}}$ and for any two objects or morphisms f, g of $\mathscr{X}$ we have $F(f \otimes g) = F(f) \otimes F(g)$.

2.5. Theorem. *Let $\mathscr{V}$ be a strict ribbon category with braiding c, twist θ, and compatible duality $(*, b, d)$. There exists a unique covariant functor $F = F_{\mathscr{V}} : \mathrm{Rib}_{\mathscr{V}} \to \mathscr{V}$ preserving the tensor product and satisfying the following conditions:*

(1) *F transforms any object $(V, +1)$ into V and any object $(V, -1)$ into V^*;*

(2) *for any objects V, W of $\mathscr{V}$, we have*

$$F(X^+_{V,W}) = c_{V,W}, \ F(\varphi_V) = \theta_V, \ F(\cup_V) = b_V, \ F(\cap_V) = d_V;$$

(3) *for any elementary v-colored ribbon graph Γ, we have $F(\Gamma) = f$ where f is the color of the only coupon of Γ.*

The functor F has the following properties:

$$F(X^-_{V,W}) = (c_{W,V})^{-1}, \ F(Y^+_{V,W}) = (c_{W,V^*})^{-1}, \ F(Y^-_{V,W}) = c_{V^*,W}, \tag{2.5.a}$$

$$F(Z^+_{V,W}) = (c_{W^*,V})^{-1}, \ F(Z^-_{V,W}) = c_{V,W^*},$$

$$F(T^+_{V,W}) = c_{V^*,W^*}, \ F(T^-_{V,W}) = (c_{W^*,V^*})^{-1}, \ F(\varphi'_V) = (\theta_V)^{-1}.$$

Theorem 2.5 plays a fundamental role in this monograph. It may be regarded from several complementary viewpoints. First of all, it yields isotopy invariants of v-colored ribbon graphs and, in particular, invariants of colored framed links in $\mathbb{R}^3$. Indeed, by definition of $\mathrm{Rib}_{\mathscr{V}}$, isotopic v-colored ribbon graphs Ω and Ω' represent the same morphism in $\mathrm{Rib}_{\mathscr{V}}$ and therefore $F(\Omega) = F(\Omega')$. As we shall see in Chapter XII these invariants form a far-reaching generalization of the Jones polynomial of links. Secondly, Theorem 2.5 elucidates the role of braiding, twist, and duality exhibiting them as elementary blocks sufficient to build up a consistent theory of isotopy invariants of links. Theorem 2.5 renders rigorous and amplifies the graphical calculus described in Section 1.6. The main new feature is the isotopy invariance of the morphisms in $\mathscr{V}$ associated to ribbon graphs. This makes Theorem 2.5 a useful tool in the study of ribbon categories. Theorem 2.5 may also be viewed as a machine extracting morphisms in $\mathscr{V}$ from ribbon graphs in $\mathbb{R}^3$.

The morphism $F(\Omega)$ associated to a v-colored ribbon graph Ω is called the operator invariant of Ω. The term "operator invariant" does not mean that $F(\Omega)$ is linear in any sense. This term is intended to remind of the following multiplicativity properties of F. Since F is a covariant functor we have

$$F(\downarrow_V) = \mathrm{id}_V, \ F(\uparrow_V) = \mathrm{id}_{V^*}, \ \text{and } F(\Omega\,\Omega') = F(\Omega)F(\Omega') \tag{2.5.b}$$

for any two composable v-colored ribbon graphs Ω and Ω'. Since F preserves the tensor product we have

(2.5.c) $$F(\Omega \otimes \Omega') = F(\Omega) \otimes F(\Omega')$$

for any two v-colored ribbon graphs Ω and Ω'. Note also that for any v-colored ribbon (0,0)-graph Ω, we have $F(\Omega) \in K = \mathrm{End}(\mathbb{1}_{\mathcal{V}})$.

The values of F on $\cup_V^-$ and $\cap_V^-$ may be computed from the formulas

(2.5.d) $$\cup_V^- = (\uparrow_V \otimes \varphi_V') \circ Z_{V,V}^+ \circ \cup_V;$$

(2.5.e) $$\cap_V^- = \cap_V \circ Z_{V,V}^- \circ (\varphi_V \otimes \uparrow_V).$$

The proof of Theorem 2.5 occupies Sections 3 and 4. The idea of the proof may be roughly described as follows. We shall use the tensor product and the composition in $\mathrm{Rib}_{\mathcal{V}}$ in order to express any ribbon graph via the ribbon graphs mentioned in the items (2) and (3) of the theorem. Such an expression allows us to define the value of F for any ribbon graph. Although every ribbon graph admits different expressions of this kind, they may be obtained from each other by elementary local transformations. To show that F is correctly defined, we verify the invariance of F under these transformations.

To demonstrate the power of Theorem 2.5 we devote the rest of Section 2 to applications of this theorem to duality, traces, and dimensions in ribbon categories. We also study the behavior of $F(\Omega)$ under simple transformations of ribbon graphs.

Up to the end of Section 2, the symbol $\mathcal{V}$ denotes a strict ribbon category. By coloring and v-coloring of ribbon graphs, we mean coloring and v-coloring over $\mathcal{V}$. As in Section 1, we shall write $\Omega \doteq \Omega'$ for v-colored ribbon graphs Ω, Ω' such that $F(\Omega) = F(\Omega')$. For instance, if $\Omega \approx \Omega'$, i.e., Ω and Ω' are isotopic, then $\Omega \doteq \Omega'$. Similarly, for a v-colored ribbon graph Ω and a morphism f in $\mathcal{V}$, we write $\Omega \doteq f$ and $f \doteq \Omega$ whenever $f = F(\Omega)$. For example, $X_{V,W}^+ \doteq c_{V,W}$, $\varphi_V \doteq \theta_V$, etc.

2.6. Applications to duality

2.6.1. Corollary. *For any object V of $\mathcal{V}$, the object V^{**} is canonically isomorphic to V.*

Proof. Consider the morphisms $\alpha_V : V \to V^{**}$, $\beta_V : V^{**} \to V$ corresponding under the functor F to the v-colored ribbon graphs in Figure 2.7 where $\mathrm{id} = \mathrm{id}_{V^*}$. Thus,

$$\alpha_V = (F(\cap_V^-) \otimes \mathrm{id}_{V^{**}})(\mathrm{id}_V \otimes b_{V^*}), \quad \beta_V = (d_{V^*} \otimes \mathrm{id}_V)(\mathrm{id}_{V^{**}} \otimes F(\cup_V^-)).$$

The argument in Figure 2.8 shows that $\beta_V \alpha_V = \mathrm{id}_V$. (We use the isotopy invariance and other properties of F established in Theorem 2.5; as an exercise the

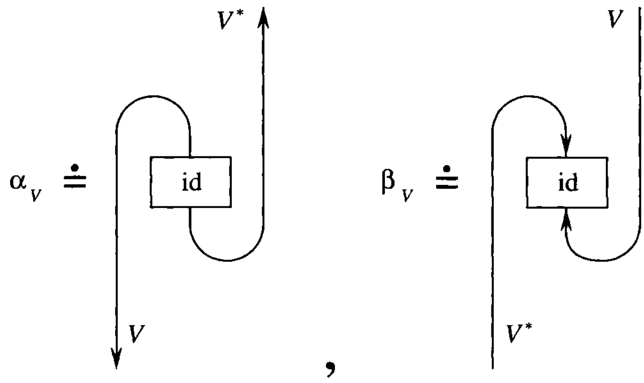


Figure 2.7

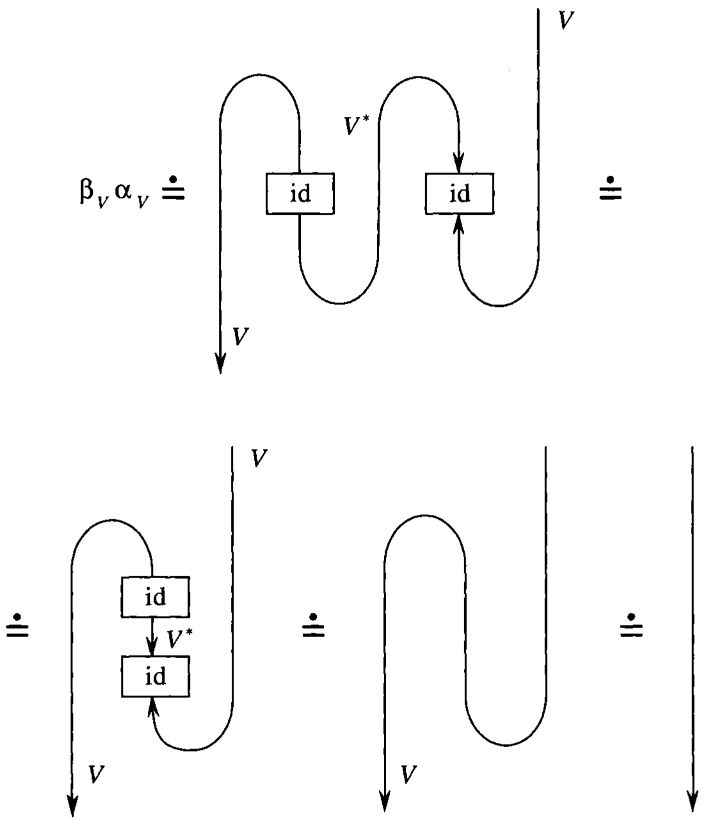


Figure 2.8

reader may rewrite the equalities in Figure 2.8 in the algebraic form.) A similar argument shows that $\alpha_V \beta_V = \mathrm{id}_{V^{**}}$.

2.6.2. Corollary. *The morphisms* $b_{\mathbb{1}} : \mathbb{1} \to \mathbb{1}^*$ *and* $d_{\mathbb{1}} : \mathbb{1}^* \to \mathbb{1}$ *are mutually inverse isomorphisms.*

Proof. The existence of isomorphism $\mathbb{1}^{**} \approx \mathbb{1}$ implies that $\mathbb{1}^*$ is isomorphic to $\mathbb{1}$. Indeed,

$$\mathbb{1}^* = \mathbb{1}^* \otimes \mathbb{1} \approx \mathbb{1}^* \otimes \mathbb{1}^{**} \approx (\mathbb{1}^* \otimes \mathbb{1})^* = (\mathbb{1}^*)^* \approx \mathbb{1}.$$

Formula (1.3.b) applied to $V = \mathbb{1}$ yields $d_{\mathbb{1}} b_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}$. This equality and the commutativity of $\mathrm{End}(\mathbb{1})$ imply that for any isomorphism $g : \mathbb{1} \to \mathbb{1}^*$, we have

$$(g^{-1} b_{\mathbb{1}})(d_{\mathbb{1}} g) = (d_{\mathbb{1}} g)(g^{-1} b_{\mathbb{1}}) = d_{\mathbb{1}} b_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}.$$

Therefore $b_{\mathbb{1}} d_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}^*}$.

2.6.3. Remark. The proof of Corollary 2.6.1 may seem to certain readers a bit light-minded and not quite convincing. In fact, the proof is complete albeit based on somewhat unusual ideas. We introduce the morphisms α_V, β_V using the functor F. Then we compute their composition using the fact that F is a covariant functor invariant under a few simple modifications of v-colored ribbon graphs. The modifications in question include isotopy and cancelling of coupons colored with the identity morphisms. The invariance of F follows from Theorem 2.5. The reader would do well to analyze the proof of Corollary 2.6.1 in detail; we shall systematically use similar arguments.

2.7. Applications to trace and dimension

2.7.1. Corollary. *Let f be an endomorphism of an object V of $\mathcal{V}$. Let Ω_f be the ribbon* $(0,0)$*-graph consisting of one f-colored coupon and one V-colored band and presented by the diagram in Figure* 2.9. *Then* $F(\Omega_f) = \mathrm{tr}(f)$.

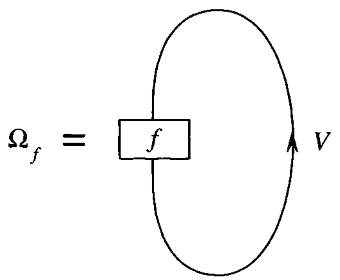


Figure 2.9

This Corollary gives a geometric interpretation of the trace of morphisms introduced in Section 1.5. Applying Corollary 2.7.1 to $f = \mathrm{id}_V$ we get $\dim(V) = F(\Omega_V)$ where Ω_V is an unknotted untwisted annulus of color V with an arbitrary direction of the core. (The annulus Ω_V is obtained from Ω_{id_V} by elimination of the coupon. This does not change the operator invariant because this coupon is colored with the identity morphism.) Note that there is an isotopy of Ω_V onto itself reversing direction of the core. This fact and the isotopy invariance of $F(\Omega_V)$ explain why $F(\Omega_V)$ does not depend on this direction.

Proof of Corollary. Let Γ_f be the v-colored ribbon (1,1)-graph presented by the diagram in Figure 1.1 with $W = V$. It is obvious that $\Omega_f \approx \cap_{\overline{V}} \circ (\Gamma_f \otimes \uparrow_V) \circ \cup_V$ where the symbol $\approx$ denotes isotopy. It follows from (2.5.e) that

$$\Omega_f \approx \cap_V \circ Z_{\overline{V},V}^{-} \circ (\varphi_V \otimes \uparrow_V) \circ (\Gamma_f \otimes \uparrow_V) \circ \cup_V.$$

Theorem 2.5 implies that $F(\Omega_f)$ is equal to the expression used to define $\mathrm{tr}(f)$.

2.7.2. Corollary (a generalization of Corollary 2.7.1). *Let Ω be a v-colored ribbon graph determining an endomorphism of a certain object of* $\mathrm{Rib}_{\mathcal{V}}$. *Let $\overline{\Omega}$ be the v-colored ribbon* $(0,0)$*-graph obtained by closing the free ends of Ω (see Figure* 2.10 *where the box bounded by broken line substitutes a diagram of Ω). Then* $\mathrm{tr}(F(\Omega)) = F(\overline{\Omega})$.

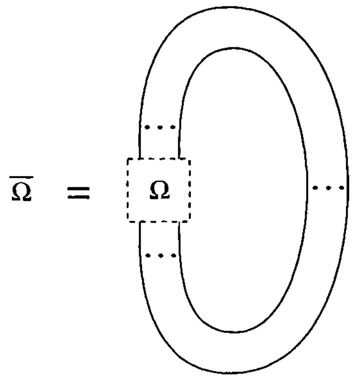


Figure 2.10

Proof. Note that the ribbon $(0,0)$-graph $\overline{\Omega}$ is obtained by connecting the top free ends of Ω to the bottom free ends of Ω in the way indicated in Figure 2.10. The v-coloring of Ω determines a v-coloring of $\overline{\Omega}$ in the obvious way. Set $V = F(\eta)$ where η is the object of $\mathrm{Rib}_{\mathcal{V}}$ which is both the source and the target of Ω. The box bounded by broken line with a diagram of Ω inside may be replaced with a coupon colored by $F(\Omega) : V \to V$ without changing the operator invariant. This yields the first equality in Figure 2.11. The second equality follows from the properties of F specified in Theorem 2.5. The isotopy in Figure 2.11 is obtained by pulling the id_V-colored coupon along the strands so that it comes close to

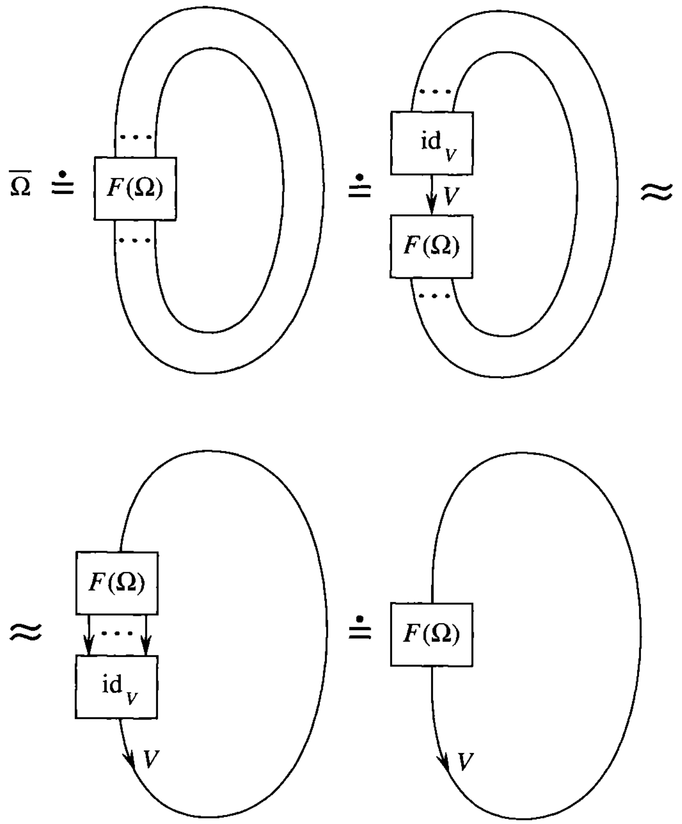


Figure 2.11

the $F(\Omega)$-colored coupon from below. The last equality in Figure 2.11 follows from the properties of F. It remains to apply Corollary 2.7.1 to deduce that $\overline{\Omega} \doteq \mathrm{tr}(F(\Omega))$.

2.7.3. Corollary. *For any objects V, W of $\mathcal{V}$, we have the equality in Figure* 2.12.

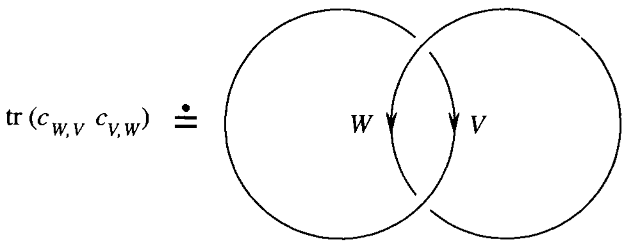


Figure 2.12

This assertion follows from Corollary 2.7.2 applied to $\Omega = X^+_{W,V}X^+_{V,W}$ and the isotopy invariance of F. The framed link presented by the diagram in Figure 2.12 is called the Hopf link with zero framing.

2.7.4. Proof of Lemma 1.5.1. The proof of the equality $\mathrm{tr}(fg) = \mathrm{tr}(gf)$ is given in Figure 2.13. The proof of the equality $\mathrm{tr}(f \otimes g) = \mathrm{tr}(f)\,\mathrm{tr}(g)$ is given in Figure 2.14. Here we use Corollary 2.7.2 and the isotopy invariance of F.

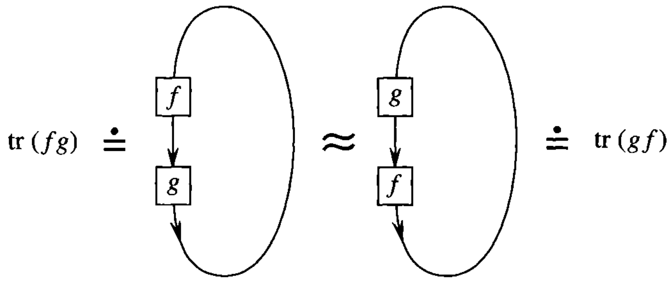


Figure 2.13

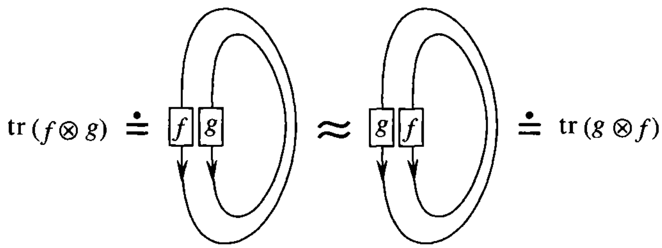


Figure 2.14

Let us show that for any $k \in K$, we have $\mathrm{tr}(k) = k$. It follows from (1.2.e), the equality $\theta_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}$, and Corollary 2.6.2 that

$$\mathrm{tr}(k) = d_{\mathbb{1}}(k \otimes \mathrm{id}_{\mathbb{1}^*})b_{\mathbb{1}} = (\mathrm{id}_{\mathbb{1}} \otimes d_{\mathbb{1}})(k \otimes \mathrm{id}_{\mathbb{1}} \otimes \mathrm{id}_{\mathbb{1}^*})(\mathrm{id}_{\mathbb{1}} \otimes b_{\mathbb{1}}) =$$

$$= k \otimes (d_{\mathbb{1}}(\mathrm{id}_{\mathbb{1}} \otimes \mathrm{id}_{\mathbb{1}^*})b_{\mathbb{1}}) = k \otimes d_{\mathbb{1}}b_{\mathbb{1}} = k \otimes \mathrm{id}_{\mathbb{1}} = k.$$

2.8. Transformations of ribbon graphs. We describe three simple geometric transformations of ribbon graphs and discuss the behavior of the operator invariant under these transformations. The first transformation is applied to an annulus component of a ribbon graph. It reverses the direction of (the core of) the annulus and replaces its color by the dual object. The second transformation is applied to an annulus component colored with the tensor product of two objects of $\mathcal{V}$; the annulus is split into two parallel annuli colored with these two objects. Finally, the third transformation is the mirror reflection of the ribbon graph with respect to the plane of our pictures $\mathbb{R} \otimes 0 \otimes \mathbb{R}$. (This reflection keeps the boundary ends of the graph.) We shall see that the first two transformations preserve the operator invariant whereas the third one involves the passage to the mirror ribbon category.

2.8.1. Corollary. *Let Ω be a v-colored ribbon graph containing an annulus component ℓ. Let Ω' be the v-colored ribbon graph obtained from Ω by reversing the direction of ℓ and replacing the color of ℓ with its dual object. Then $F(\Omega') = F(\Omega)$.*

Proof. Denote the color of ℓ by V. Choose a small vertical segment of ℓ directed upwards and replace it by the composition of two coupons Q_1 and Q_2 both colored with id_{V^*} (see Figure 2.15 where the distinguished (bottom) bases of Q_1 and Q_2 are the lower horizontal bases). This transformation does not change the operator invariant of Ω. Now pulling the coupon Q_1 along ℓ we deform the graph in $\mathbb{R}^3$ so that at the end Q_1 comes close to Q_2 from below. Since the colors of Q_1 and Q_2 are the identity endomorphisms of V^* we may eliminate Q_1 and Q_2 in this final position without changing the operator invariant. This yields Ω'. Hence $F(\Omega') = F(\Omega)$.

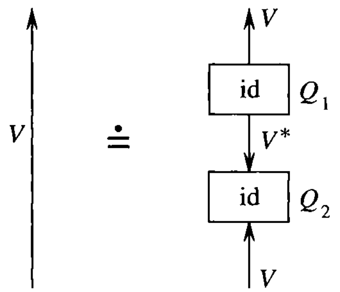


Figure 2.15

2.8.2. Corollary. *For any object V of $\mathcal{V}$, we have* $\dim(V^*) = \dim(V)$.

Proof. The oriented trivial knot is isotopic to the same knot with the opposite orientation. Therefore Corollary 2.8.1 and the remarks following the statement of Corollary 2.7.1 imply Corollary 2.8.2.

2.8.3. Corollary. *Let Ω be a v-colored ribbon graph containing an annulus component ℓ of color $U \otimes V$ where U and V are two objects of $\mathcal{V}$. Let Ω' be the v-colored ribbon graph obtained from Ω by cutting ℓ off along its core and coloring two newly emerging annuli with U and V. Then $F(\Omega) = F(\Omega')$.*

Proof. The idea of the proof is the same as in the proof of the previous corollary. Take a small vertical segment of ℓ directed downwards and replace it by two coupons as in Figure 2.16 where id $= \mathrm{id}_{U\otimes V}$. It is obvious that this modification does not change the operator invariant. Now, pushing the upper coupon along ℓ until it approaches the lower coupon from below we deform our ribbon graph in the position where we may cancel these two coupons. This does not change the operator invariant and results in Ω'. Hence $F(\Omega) = F(\Omega')$.

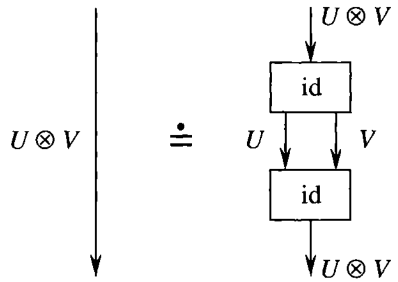


Figure 2.16

2.8.4. Corollary. *Let Ω be a v-colored ribbon graph over $\mathcal{V}$. Let $\overline{\Omega}$ be its mirror image with respect to the plane $\mathbb{R} \times 0 \times \mathbb{R}$. Then*

$$F_{\overline{\mathcal{V}}}(\overline{\Omega}) = F_{\mathcal{V}}(\Omega).$$

Note that to get a diagram of $\overline{\Omega}$ from a diagram of Ω we should simply trade all overcrossings for undercrossings. For instance, the mirror images of $X^+_{V,W}, Y^+_{V,W}, Z^+_{V,W}, T^+_{V,W}$ are $X^-_{V,W}, Y^-_{V,W}, Z^-_{V,W}, T^-_{V,W}$ respectively.

Proof of Corollary. Consider the covariant functor $G : \mathrm{Rib}_{\mathcal{V}} \to \mathcal{V}$ which coincides on the objects with $F_{\mathcal{V}}$ and transforms the morphism represented by a v-colored ribbon graph Ω into $F_{\overline{\mathcal{V}}}(\overline{\Omega})$. (We are allowed to regard $F_{\overline{\mathcal{V}}}(\overline{\Omega})$ as a morphism in the category $\mathcal{V}$ because $\overline{\mathcal{V}}$ and $\mathcal{V}$ have the same underlying monoidal category.) It is straightforward to see that G satisfies conditions (1) - (3) of Theorem 2.5. In particular,

$$G(X^+_{V,W}) = F_{\overline{\mathcal{V}}}(X^-_{V,W}) = (\overline{c}_{W,V})^{-1} = c_{V,W} = F_{\mathcal{V}}(X^+_{V,W})$$

and

$$G(\varphi_V) = F_{\overline{\mathcal{V}}}(\varphi'_V) = (\overline{\theta}_V)^{-1} = \theta_V = F_{\mathcal{V}}(\varphi_V).$$

The uniqueness in Theorem 2.5 implies that $G = F_{\mathcal{V}}$. This yields our claim.

2.8.5. Corollary. *The dimensions of any object of $\mathcal{V}$ with respect to $\mathcal{V}$ and $\overline{\mathcal{V}}$ are equal.*

This follows from Corollary 2.8.4 since the mirror image of a plane annulus is the same annulus.

2.9. Exercises. 1. Let Ω be a v-colored ribbon graph containing an annulus component. Let Ω' be the v-colored ribbon graph obtained from Ω by replacing the color of this component with an isomorphic object. Show that $\Omega \doteq \Omega'$. What is the natural analogue of this assertion for bands? What are the analogues of Corollaries 2.8.1 and 2.8.3 for bands?

2. Let Ω be a v-colored ribbon graph containing an annulus component or a band of color $\mathbb{1}$. Let Ω' be the v-colored ribbon graph obtained from Ω by eliminating this annulus (resp. band). Show that $\Omega \doteq \Omega'$.

3. Show that for any three objects U, V, W of a ribbon category, the formulas $f \mapsto (\mathrm{id}_V \otimes F(\cap_W^-))(f \otimes \mathrm{id}_{W^*})$ and $g \mapsto (g \otimes \mathrm{id}_W)(\mathrm{id}_U \otimes F(\cup_W^-))$ establish mutually inverse bijective correspondences between the sets $\mathrm{Hom}(U, V \otimes W)$ and $\mathrm{Hom}(U \otimes W^*, V)$. Write down similar formulas for a bijective correspondence between $\mathrm{Hom}(U \otimes V, W)$ and $\mathrm{Hom}(V, U^* \otimes W)$.

4. Show that if $\theta_V = \mathrm{id}_V$ for all objects V of $\mathcal{V}$ then for any colored ribbon graph Ω consisting of m annuli, we have $F(\Omega) = \prod_{i=1}^m \dim(V_i)$ where $V_1, ..., V_m$ are the colors of these annuli. This applies, for example, to the ribbon category constructed in Section 1.7.1.

5. Let $\mathcal{V}$ be the ribbon category $\mathcal{V}(G, K, c, \varphi)$ constructed in Section 1.7.2. Use formulas (2.5.d), (2.5.e) to show that for any $g \in G$, we have $F(\cup_g^-) = \varphi(g)$ and $F(\cap_g^-) = \varphi(g)$. Deduce from these equalities that $\dim(g) = \varphi(g)$ for any $g \in G$. Show that for a framed m-component link $L = L_1 \cup ... \cup L_m$ whose components are colored with $g_1, ..., g_m \in G$ respectively, we have

$$F(L) = \prod_{1 \leq j < k \leq m} (c(g_j, g_k)\, c(g_k, g_j))^{l_{jk}} \times \prod_{j=1}^{m} c(g_j, g_j)^{l_j} \varphi(g_j)^{l_j+1}$$

where $l_{jk} \in \mathbb{Z}$ is the linking number of L_j and L_k, and $l_j \in \mathbb{Z}$ is the framing number of L_j.

6. Let V be an object of a ribbon category $\mathcal{V}$ such that any endomorphism of V has the form $k \otimes \mathrm{id}_V$ for certain $k \in K = \mathrm{End}(\mathbb{1}_{\mathcal{V}})$. Let Ω be a v-colored ribbon $(0,0)$-graph containing an annulus of color V. Show that $F(\Omega)$ is divisible by $\dim(V)$ in the semigroup K. (Hint: present Ω as the closure of a v-colored ribbon (1,1)-graph which is an endomorphism of $(V, 1)$.)

7. Show that if duality in a strict monoidal category $\mathcal{V}$ is compatible with a braiding and a twist then the square of the duality functor $(V \mapsto V^*, f \mapsto f^*) : \mathcal{V} \to \mathcal{V}$ is canonically equivalent to the identity functor $\mathcal{V} \to \mathcal{V}$ (cf.

Exercise 1.8.2). Show that for any endomorphism f of any object of $\mathcal{V}$, we have $\operatorname{tr}(f^*) = \operatorname{tr}(f)$.

3. Reduction of Theorem 2.5 to lemmas

3.0. Outline. The functor F may be regarded as a "linear representation" of the category $\mathrm{Rib}_{\mathcal{V}}$. This point of view allows us to appeal to the standard technique of group theory: in order to define a linear representation of a group one assigns matrices to generators and checks definining relations. Following this line we shall introduce generators and relations for $\mathrm{Rib}_{\mathcal{V}}$ and use them to construct F.

The material of Sections 3 and 4 will not be used in the remaining part of the book and may be skipped without harm for what follows. Still, the author finds the arguments given in these two sections beautiful and instructive in themselves.

3.1. Generators for $\mathrm{Rib}_{\mathcal{V}}$. Our immediate aim is to describe the category of v-colored ribbon graphs $\mathrm{Rib}_{\mathcal{V}}$ and its subcategory of colored ribbon tangles in terms of generators and relations.

We say that a family of morphisms in a strict monoidal category $\mathcal{X}$ generates $\mathcal{X}$ if any morphism in $\mathcal{X}$ may be obtained from these generators and the identity endomorphisms of objects of $\mathcal{X}$ using composition and tensor product. A system of relations between the generating morphisms is said to be complete if any relation between these morphisms may be deduced from the given ones using the axioms of strict monoidal category. For a more detailed discussion of generators and relations in monoidal categories, see Section 4.2.

Recall the morphisms in $\mathrm{Rib}_{\mathcal{V}}$ introduced at the end of Section 2.3.

3.1.1. Lemma. *The colored ribbon tangles*

$$X^{\nu}_{V,W},\ Z^{\nu}_{V,W},\ \varphi_V,\ \varphi'_V,\ \cup_V,\ \cap_V \tag{3.1.a}$$

where V, W run over objects of $\mathcal{V}$ and ν runs over $+1, -1$ generate the category of ribbon tangles. The same ribbon tangles together with all elementary v-colored ribbon graphs generate $\mathrm{Rib}_{\mathcal{V}}$.

Proof. To prove the lemma we need the notion of a generic tangle diagram. Let $D \subset \mathbb{R} \times [0, 1]$ be a diagram of a ribbon tangle. By the height function on D, we mean the projection $\mathbb{R}\times[0, 1] \mapsto [0, 1]$ restricted to D. By an extremal point of D, we mean a point of D lying in $\mathbb{R} \times (0, 1)$ (i.e., distinct from the end points of D) such that the height function on D attains its local maximum or local minimum in this point. By singular points on D, we mean extremal points and crossing points of D. We say that D is generic if its extremal points are distinct from its crossing points, the singular points of D are finite in number and lie on different levels of the height function, and the height function is non-degenerate in all extremal points. The last condition means that in a neighborhood of any extremal point the diagram D looks like a cup or a cap, i.e., like the graph of the function x^2 or $-x^2$ near $x = 0$.

It is obvious that a small deformation transforms any tangle diagram into a generic tangle diagram. Therefore every ribbon tangle may be presented by a generic tangle diagram.

Take an arbitrary ribbon tangle and present it by a generic diagram $D \subset \mathbb{R} \times [0, 1]$. Consider the boundary lines of the strip $\mathbb{R} \times [0, 1]$ and draw several parallel horizontal lines in this strip so that between any two adjacent lines lies no more than one singular point of D. It is clear that the part of D lying between such adjacent lines represents the tensor product of several identity morphisms and one morphism from the family of morphisms drawn in Figures 2.5 and 2.6 (except φ, φ'). The ribbon tangle presented by D is decomposed in this way in a composition of such tensor products. To prove the first assertion of the lemma it remains to express the tangles drawn in Figures 2.5 and 2.6 via the tangles (3.1.a) and the "identity" tangles $\uparrow_V$, $\downarrow_V$. Such expressions are provided by (2.5.d), (2.5.e), and

(3.1.b) $$Y^{\nu}_{V,W} = (\cap_V \otimes \downarrow_W \otimes \uparrow_V)(\uparrow_V \otimes X^{\nu}_{W,V} \otimes \uparrow_V)(\uparrow_V \otimes \downarrow_W \otimes \cup_V),$$

(3.1.c) $$T^{\nu}_{V,W} = (\cap_V \otimes \uparrow_W \otimes \uparrow_V)(\uparrow_V \otimes Y^{\nu}_{W,V} \otimes \uparrow_V)(\uparrow_V \otimes \uparrow_W \otimes \cup_V)$$

where $\nu = \pm 1$. In the last formula we substitute (3.1.b) to get an expression for $T^{\nu}_{V,W}$ via the generators. (The reader is urged to draw the corresponding pictures.)

The second assertion of Lemma is proven similarly: in addition to crossing points and local extrema on a diagram we should single out the coupons and apply the same argument.

3.2. Relations between generating tangles. Here is a list of fundamental relations between the tangles (3.1.a):

(3.2.a) $$(\downarrow_W \otimes X^+_{U,V})\,(X^+_{U,W} \otimes \downarrow_V)\,(\downarrow_U \otimes X^+_{V,W}) =$$

$$= (X^+_{V,W} \otimes \downarrow_U)\,(\downarrow_V \otimes X^+_{U,W})\,(X^+_{U,V} \otimes \downarrow_W),$$

(3.2.b) $$\downarrow_V = (\downarrow_V \otimes \cap_V)\,(\cup_V \otimes \downarrow_V),$$

(3.2.c) $$\uparrow_V = (\cap_V \otimes \uparrow_V)\,(\uparrow \otimes \cup_V),$$

(3.2.d) $$X^-_{V,W} = (X^+_{W,V})^{-1},$$

(3.2.e) $$\varphi'_V = (\varphi_V)^{-1},$$