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HIGH-DIMENSIONAL NONLINEAR DIFFUSION STOCHASTIC PROCESSES

Modelling for Engineering Applications

Yevgeny Mamontov
Magnus Willander

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To our families

Preface

Almost every book concerning diffusion stochastic processes (DSPs) is devoted to the problems arising *inside* stochastic theory in the course of its logical development. The corresponding results serve inherent needs of mathematics. This way of research is related to what is usually called *pure* mathematics.

Those of its achievements which *in principle* allow quantitative treatments can also be applied to the stochastic problems arising *outside* pure mathematics. Such applications always assume practical implementation. They are addressed to scientists or engineers, both mathematicians and nonmathematicians, who are not necessarily specialists in stochastic theory. In so doing, the forms of the recipes granted by pure mathematics are not discussed since the main attention is paid to the corresponding numerical techniques.

However, the advantage to allow quantitative analysis in principle does not always mean to allow it in reality. In reality one has to deal with:

(1) people who work in dissimilar sciences or branches of engineering and perhaps do not consider pure mathematics as the main field of their activities; some of the above recipes may be recognized by these people as not very transparent;
(2) tools such as computer systems or software environments that, alas, are not of unlimited capabilities; indeed, great efforts are in

many cases made to cram the above recipes (not always aimed at any practical applications) into tight frames of the existing tools.

These circumstances constitute the problems which are out of a customary stream of pure mathematics and, in view of this, form a new and very stimulating field of research.

The present book is intended (see Section 1.13) to facilitate solution of these two problems in the field of nonlinear diffusion stochastic processes (DSPs) of a large number of variables. More precisely, the book considers DSPs in Euclidean space of high dimension (much greater than a few units) with the coefficients nonlinearly dependent on the space coordinates. Many complex stochastic systems in science and engineering lead to such DSPs.

As is well-known, DSPs are closely related to Itô's stochastic differential equations (ISDEs). This fact explains why DSP theory is usually interpreted from the viewpoint of these eqautions. However, such treatment is not only unnecessary but also presumes reader's expertise in ISDEs. To eliminate this complication means to contribute to solution of the above problem (1). Accordingly, the present book derives (see Chapters 2–4) analysis of high-dimensional DSPs only from the well-known expressions for drift vector and diffusion matrix of the processes and from Kolmogorov's forward equations. No techniques directly related to ISDE theory are involved. The corresponding prerequisites for reading (Section 1.1) are merely awareness of some basic facts associated with probability theory. The outcome of the purely DSP treatment is two-fold:

- compared with the ISDE-based alternative, the developed treatment is simpler, compact and uniform; it can be more willingly accepted by nonspecialists in stochastic integral and differential calculuses, in particular, by nonmathematicians;
- the derived representations have not been revealed in stochastic theory yet and are also of purely mathematical interest; they provide a deeper insight into behavior of the basic characteristics of DSPs that can contribute to a greater success of the subsequent analyses.

Regarding the above problem (2), i.e. to better adapt the DSP-theory results to limited power of majority of computing systems, the present book concentrates on the analytical part of a combined, analytical-numerical approach. The main advantage of this approach is that it presents a reasonable compromise between accuracy of the applied models and complexity of their numerical analysis.

The above analytical part is constructed (Chapters 2-4) with the help of the purely DSP treatment. It grants the benefit to derive such analytical approximations which allow for the nonlinearities of the DSP coefficients and, in conjunction with proper, parallel simulation techniques, can lead to realistic computing expenses even if the process is of a high dimension. These features enable one to solve much wider family of high-dimensional stochastic problems by means of numerical method and make this analysis more accessible for scientific and engineering community.

The present book concerns many aspects to better elucidate the involved notions and techniques. In particular, it includes a separate chapter (Chapter 3) devoted to nonstationary invariant processes. This topic is of importance in both qualitative and quantitative analyses. Nevertheless, it is regrettably missed out in most of the works on applications of DSPs.

Another special feature of the present book is the chapter (Chapter 5) on modelling non-Markov phenomena with various ISDEs and their connection to high-dimensional DSPs. This connection is provided by means of the well-known stochastic adaptive interpolation (SAI) method. The chapter equips this method with bases of function Banach spaces and shows how it can involve the proposed analytical-numerical treatment to approximately solve Itô's stochastic partial (integro-)differential equations.

The specific examples concerning the above chapter are considered in Chapter 6. It is devoted to Itô's stochastic partial differential equations of semiconductor theory which are based on the fluid-dynamics model. The first part of the chapter emphasizes the capabilities of common, nonrandom semiconductor equations to describe the electron (or hole) fluid not only in macroscale but also in mesoscale and microscale domains. This topic is considered in connection with a certain limit case of a proper random walk. The second part of the chapter discusses Itô's stochastic generalization of these equations for macroscale and weakly mesoscale domains and summarizes some of the corresponding results on noise in semiconductors. This also includes related topics for future development.

Chapter 7 focuses on the distinguishing features of engineering applications. A combined, analytical-numerical approach, its advantages and disadvantages as well as the connection to common analytical DSP results are discussed in Chapter 8.

For reader's convenience, the book includes the introductory chapter (Chapter 1) that lists some basic facts and methods of DSP theory and some of the related practical problems. This chapter leads to the detailed formulation (Section 1.13) of the purpose of the book.

The book includes many appendixes. They present the proofs of the lemmata and theorems, descriptions of the examples and the technical details which are needed in the course of the consideration, can contribute to better understand some stochastic features of the nonrandom models or can be helpful in practical implementation of the described methods.

The theorems are numbered with the two-position numbers in the form (K.L) where L is the cardinal number of the theorem in Chapter K. The definitions, lemmata, corollaries and remarks are numbered similarly.

The formulas in the book are numbered with the two- or three-position numbers in the forms (K.L) or (K.M.L) respectively. In so doing, position L is the cardinal number of the formula in Chapter K or, if the chapter is divided into sections, in Section K.M, i.e. the Mth section of Chapter K. The end of Index includes the list of all the definitions, lemmata, theorems, corollaries and remarks.

The application field which the book aims the developed models and methods at can include different subjects but can be formulated very briefly. This field is complex systems, no matter which specific science or engineering they are associated with. The book considers various aspects related to applied research. For instance, it:

- stresses the mathematical reading and practical meaning of the signal-to-noise ratio (Appendix B);
- explains how a common, nonrandom equation for fluid concentration includes not very simple stochastic phenomena (Appendix F);
- presents a *fully time-domain*, approximate analytical model to evaluate the particle-velocity covariance for a fairly general class of fluids (Section 4.10 and Appendix C);
- shows how to describe the long, non-exponential "tails" of this covariance by means of the ISDE with a *nonlinear* friction (Section 4.10.3);
- derives the nonrandom model which suggests an explanation of stochastic resonance and other *stochastics-induced* effects in the mean responses of the systems governed by *nonlinear* DSPs (Section 2.4).

These examples demonstrate not only the capabilities of the developed techniques but also emphasize the usefulness of the complex-system-related approaches to solve some problems which have not been solved with the traditional, statistical-physics methods yet. From this veiwpoint, the book can be regarded as a kind of complement to such books as "*Introduction to the Physics of Complex Systems. The Mesoscopic Approach to Fluctuations, Nonlinearity and Self-Organization*" (Serra, Andretta, Compiani and Zanarini; 1986), "*Stochastic Dynamical Systems. Concepts, Numerical Methods, Data Analysis*" (Honerkamp, 1994), "*Statistical Physics: An Advanced Approach with Applications*" (Honerkamp, 1998) which deal with physics of complex systems, some of the corresponding analysis methods and an innovative, stochastics-based vision of theoretical physics.

Who should read this book? We believe that any reader whose work or study concerns nonlinear high-dimensional stochastic systems will find in it something that could really help and make the painstaking but interesting job more enjoyable and fruitful. To be specific in this issue, we would point out the following groups of readers:

- nonmathematicians (e.g.,theoretical physicists, engineers in industry, specialists in models for finance or biology, computing scientists);
- mathematicians;
- undergraduate and postgraduate students of the corresponding specialties;
- managers in applied sciences and engineering dealing with the advancements in the related fields;
- any specialists who use diffusion stochastic processes to model high-dimensional (or large-scale) nonlinear stochastic systems.

and other communities who are interested in the topic of the book.

To what extent is this book suitable and useful for you personally? The best way to get the answer is to look through the "Contents" and Section 1.1 "Prerequisites for Reading". They present the information that helps you to make your decision.

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