International Series in Operations Research & Management Science

Joseph C. Paradi
H. David Sherman
Fai Keung Tam

Data Envelopment Analysis in the Financial Services Industry

A guide for practitioners and analysts working in Operations Research using DEA

Springer

International Series in Operations Research & Management Science

Volume 266

More information about this series at http://www.springer.com/series/6161

Joseph C. Paradi • H. David Sherman
Fai Keung Tam

Data Envelopment Analysis in the Financial Services Industry

A Guide for Practitioners and Analysts Working in Operations Research Using DEA

Springer

Joseph C. Paradi
University of Toronto
Toronto, ON, Canada

H. David Sherman
Northeastern University
Boston, MA, USA

Fai Keung Tam
University of Toronto
Toronto, ON, Canada

ISSN 0884-8289 ISSN 2214-7934 (electronic)
International Series in Operations Research & Management Science
ISBN 978-3-319-69723-9 ISBN 978-3-319-69725-3 (eBook)
https://doi.org/10.1007/978-3-319-69725-3

Library of Congress Control Number: 2017955958

Printed on acid-free paper

This Springer imprint is published by Springer Nature
The registered company is Springer International Publishing AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

To our significant others: Monika, Linda, and Bernice; David's supportive daughters Amanda and Caroline; and Joseph's sons Joseph and David and grandchildren Andrew, Laura, and Sophie

Foreword

I am extremely pleased that the authors have written this book and honored that they have invited me to develop the foreword.

I have known Joe Paradi and David Sherman for a number of years and have followed their pioneering research from the beginning. Joe was the first to utilize extensive data visualization to communicate DEA results to managers, while David wrote the first introductory monograph explaining DEA for the service sector. Both have extensive consulting experience and managerial expertise which produce a unique and valuable perspective. Over the years, they have developed separate impressive research agendas. I am very pleased that they have joined forces with Fai Keung Tam to produce this book. We have had many discussions of the critical need for such a book collecting together and showcasing studies of managerial importance. The result should help the reader better appreciate the power of DEA as a novel approach for organizing and analyzing data to produce valuable insight.

As mentioned in their introduction, there has been a host of DEA-related articles produced in the past 40 years. The DEA bibliography that I maintain now contains around 15,000 books, dissertations, and articles published since 1978. Unfortunately, the majority of these articles are not particularly useful. Many are a simple study of a specific industry in a single country at one point in time for which the results simply state the relative efficiency scores for a list of DMUs. Such articles are not valuable in that they are a simple ordering of units and do not provide helpful insight for managers such as trends, comparisons across regions, organizational subgroups or ownership types, multinational comparisons, etc. In short, the explanatory power is small frequently due to the shortage of temporal data, failure to perform a thorough analysis across multiple models, model extensions, and various subsets of the data and/or shortcomings of the experimental design.

This book seeks to address this problem by showcasing articles from the financial services area that describe innovative approaches and novel applications that provide insight and uncover transferable best practices. Of course the models, approaches, and advice while stated in the context of financial services are easily applicable to other industry sectors.

My hope is that DEA researchers will familiarize themselves with these compelling applications and approaches and heed the authors' guidance and advice. Hopefully this will result in a significant increase in the number of useful DEA articles for which rigorous analysis produces valuable insight and directly impacts managerial practice. Such an advancement will enhance the field and more fully realize the potential of the DEA methodology.

Lawrence M. Seiford
Department of Industrial and Operations Engineering
The University of Michigan
Ann Arbor, MI, USA

Acknowledgments

Once a person writes a book, s/he learns the hard lesson that it takes several times more effort than anticipated, longer than planned and the completion almost becomes an obsession. Of course, aside from the authors of the book, a lot of others make contributions, some more, some less, but all are essential to success.

First, we would like to thank the owners of the copyrighted materials they so graciously allowed us to use and include in this book. All good contributions to science are based on the work that has been done by many others in the far as well as in the recent past.

Aside from the use of copyrighted materials from external sources, we made use of a substantial amount of research results and work completed or being worked on by our own students. These outstanding young women and men form the foundation of the future in not only DEA but all aspects of our society. We appreciate their enthusiasm in helping us with this book. They deserve much of the credit for the ideas, development, and progress in the application of DEA to real-life problems. Here they are and our postdoctoral fellows:

Burçu Anadol	Parisa H. Ardehali	Maryam Badrizadeh
Barak Edelstein	Allison Hewlitt	Angela Tran Kingyens
Alex E. LaPlante	Denise McEachern	Elizabeth Min
Peter Pille	Stephen Rouatt	Paul C. Simak
Shabnam Sorkhi	Taraneh Sowlati	Niloofar Tochaie
Sandra A. Vela	D'Andre Wilson	Tracy Yang
Zijiang Yang		

Postdoctoral fellows: Mette Asmild, Dan Rosen, Claire Schaffnit, Xiaopeng Yang, and Haiyan Zhu

We also thank all who suggested ways to do things, provided examples of how to view real-world problems, and added the "reality" factor to the work we reported on. We thank dozens of professionals who collaborated with us in the work, without whom the underlying research and therefore this book could not have been completed. Among these stands out David Paradi who is a master at using PowerPoint and has contributed his knowledge and enthusiasm to the production of figures we present here, and many other technical issues.

A special thank you is due to Professor Joe Zhu who suggested to us that a book like this was needed and then answered all our questions. He is one of today's most respected authors and authorities on DEA. Very much is owed to our late friend, Prof. W.W. (Bill) Cooper, who was one of the creators of DEA and was first to introduce us to the boundless problem-solving capabilities of this excellent tool.

And last, but not least, we thank our better halves Monika Paradi, Linda Sherman, and Bernice Cheng for their patience and even encouragements while they were neglected during the creation of this book.

Contents

List of Figures

List of Tables

Introduction

Data envelopment analysis was first titled with this name in the paper by Charnes, Cooper, and Rhodes in 1978. The initials DEA have since been widely adopted. The concept was previously exposed in Farrell's seminal paper (1957): "The measurement of productive efficiency." Farrell did not have the power of modern computing equipment at his disposal, so the development of practical applications was not feasible in a practical sense. But time passed and technology developed so that Farrell's work became possible to apply to complex problems with multiple inputs and outputs. Linear programming capabilities allowed the DEA models to be used for varied problems. Running DEA often required rerunning a linear program thousands of times, a capability that was not readily available in the 1950s. Today, running numerous linear programming iterations required for DEA can be done on the average personal computer by simply using DEA custom-coded programs or even Microsoft Excel.

Slowly, researchers in operational research and economics began to apply DEA to their problems. With few exceptions, their primary goal was to extend the theoretical foundations of the science and report this in traditional academic refereed journals in management science, economics, social science, and mathematics. As more researchers became involved in looking at DEA as a fruitful approach to management and economic problems and their works were published, the literature grew, at first slowly and in recent years quite rapidly. While in the early days it was possible to keep up with the new papers as they appeared (e.g., Seiford 1997; Emrouznejad et al. 2008), this is now essentially impossible as it would take a person working full time just to assemble the bibliography. The number of books alone now published is around 100 and growing. The DEA technology is now well established but still developing, and relatively small theoretical additions, extensions, and refinements continue to be reported in the academic literature. One of the best sources of the most up-to-date information on DEA is found at A. Emrouznejad's DEA Zone on the web (2017): www.deazone.com.

However, the major challenge and unfinished DEA work is, in our view, that only a small portion of the published works deal with applications of DEA to real-life problems and even fewer result in production systems making use of DEA. The reader might challenge this assertion, so let us clarify. While most recent papers used real data obtained from credible sources, such as the OECD, national statistics from agriculture to retirement homes, and the financial and economic data sources, studies based on such data do not enable managers to obtain directions on how to enact policy or improve practice in their businesses or other organizations. Even when there is the potential to apply the DEA findings to real operating organization datasets, the results of the analyses are published without pursuing the application to generate the potential benefits. While there are examples where the results have been applied and the positive and negative results are reported, these papers reflect an incredibly small fraction of the total DEA published literature. There are applications that have been successful that have not been published, and while we cannot know the universe of the works not published, discussions with academics and end users of DEA suggest that these unpublished applications are not likely to be very large in number. Two fields that stand out in these studies are health care and banking where hundreds of papers were written over the past couple of decades, but with very few being of practical use to the people who operate these institutions.

The early focus of DEA was applying it to units in any organization that have control over their activities, and where there is some manager that assesses performance of each unit and makes decisions about how the unit operates in an effort to improve its outcomes. The term adopted for these operating units was *decision-making units*, or DMUs. These initials, DMUs, are well understood by the DEA community, but this is not a term that is naturally found or used in business, government, or other organizations. The terminology in itself may be sufficiently arcane and unfamiliar to potential users that it may have contributed to the slow adoption of DEA. The current use of DEA continues to heavily focus on understanding and improving the performance of the defined DMUs, but has also broadened to recognize DEA's ability to identify relationships in complex operating data that offer new insights into the way organizations operate and other paths to manage performance (Sherman and Zhu 2013).

The definitions of what a DMU is determine the usability of the results. For economists, the aggregate data is useful when they advise governments on policy or evaluate the national or international health of certain sectors of interest. The DMU may be defined as a political unit, country, industry, etc. But useful direction for the managers of units such as bank branches, hospital departments, farms, retirement homes, etc. is seldom provided, yet this is where real operating benefits can be achieved. For example, when a study is conducted on the efficiency, productivity, or effectiveness of the banking industry, the outcomes for each bank (the DMUs in the models) offer no implementable findings as the data is aggregated and applies to the DMU as a whole. Of course, the outcome of such a study may well be useful for the regulator or government evaluation of the health of the industry and in identifying regulatory policies that would improve overall productivity. A concrete example of DEA being applied to help regulate an industry is the utilities sector, where it has been used to manage electricity producers in Europe and Brazil (Agrelle et al. 2005).

When the focus is on DMUs that are finite operating units such as bank branches, clinics, physicians, hospitals, nursing homes, and focused services in health care, DEA provides, in addition to an assessment of the DMUs, insights that can allow a manager to directly adjust methods of operations. These adjustments can provide the opportunity to measurably improve the performance of the DMUs analyzed with DEA.

This book is intended to address the challenge of how to apply the DEA technology to data, where the data is relevant and detailed enough to allow results to be useful to the managers by implementing the outcomes from the study to improve the performance of their organizations. In other words, we look at the practitioners' problem of applying improvements to the businesses or institutions where the benefits are directly received by the owners, employees, and/or customers of the firm. Of course, the entire firm benefits from the individual improvements. For example, an analysis of a retail chain store or franchising operation where managers do have the power to implement the improvements suggested by a DEA analysis could result in lower costs for individual operating units (the DMUs in this type of study), improving profitability of these units and thus an augmentation of the system-wide success and attractiveness of owning one of the franchised units.

Our intent and objective is to provide any reader of this book a set of useful approaches and techniques which they can apply and, if done as suggested in this volume, would enable the reader to improve their firm's performance (or that of their client firm if they are consulting for them). However, there are many sectors in a large economy and no single book can cover them all. Therefore, we restricted ourselves to the financial sector where there are a number of studies published examining the actual performance level of the firm and where the firm should go to reap the benefits of the study. Perhaps it would be appropriate to see this book as a *how-to* manual where the practitioner or analyst can find a study that relates to their problem, often directly, while other times they may find an example where there are similarities to their organization but which requires some adaptation to be effective.

References: Introduction

Agrelle, P.J., Bogetoft, P., Jørgen, T: DEA and dynamic yardstick competition in Scandinavian electricity distribution. J. Product. Anal. **23**(2), 173–201 (2005)

Charnes, A., Cooper, W.W., Rhodes, E.: Measuring the efficiency of decision making units. Eur. J. Oper. Res. **2**(6), 429–444 (1978)

Emrouznejad, A.: Ali Emrouznejad's data envelopment analysis. http://www.deazone.com (2017). Accessed 30 Nov 2016

Emrouznejad, A., Parker, B.R., Tavares, G.: Evaluation of research in efficiency and productivity: a survey and analysis of the first 30 years of scholarly literature in DEA. Socio-Econ. Plan. Sci. **42**(3), 151–157 (2008)

Farrell, M.J.: The measurement of productive efficiency. J. R. Stat. Soc. **120**(3), 253–290 (1957)
Seiford, L.M.: A bibliography for data envelopment analysis (1978–1996). Ann. Oper. Res. **73**, 393–438 (1997)
Sherman, H. D., Zhu, J.: Analyzing performance of service organizations. MIT Sloan Manag. Rev. **54**(4), 37–44 (2013)

Part I
Data Envelopment Analysis, in Brief, with Little Math!

Introduction

In Part I of this book we provide an overview of DEA but without a lot of mathematics, except to allow the explanations to make sense. Our intent is to allow the reader to assess the technology and understand it well enough to delve into whatever details he or she feels necessary to their needs. This part also provides a brief survey and summary of some of the large body of published DEA studies on banking and other financial services.

Chapter 1
DEA Models Overview

We begin with the basic DEA Models and some useful extensions (although we expect that some will see it as too much while others as too little). While we promised to minimize the mathematics, some are, unfortunately, unavoidable. We have excluded any specific discussion of the underlying linear programming (LP) mathematics that drives DEA, and while some general understanding of this is helpful for understanding the academic literature, it is not needed to understand the benefits and ways to apply DEA.

The next issue here is how to select what are "inputs" and what are "outputs". One would assume that this is easy since whatever is used in the production model is an input and what is produced is the output. But there are some issues, such as undesirable outputs (e.g. bad loans) and inputs where we might want more (e.g. deposits in a bank branch). To make matters more confusing, some measures may well be used as inputs in some models while outputs in others – such as bank deposits. We also address some data issues in this part.

Model formation is another subject fraught with controversy and we point to some issues and suggest ways to address these problems.

Finally, we provide a brief history of DEA and its development and sketch out the DEA models family tree to show how things connect together from the Charnes et al. (1978) model to some of the more sophisticated models that have been developed since that seminal paper saw the light of day. Some of the milestone applications of DEA in the financial services are also provided.

J.C. Paradi et al., *Data Envelopment Analysis in the Financial Services Industry*, International Series in Operations Research & Management Science 266,
https://doi.org/10.1007/978-3-319-69725-3_1

Basic DEA Models

Introduction

Data Envelopment Analysis (DEA) is an example of a frontier approach. Frontier approaches identify and assess the areas or examples of best performance or best practice within the sample, i.e. those located on the "frontier". The frontier identified by DEA suggests the best performance within the group of operating units being evaluated and does not promise or even suggest that these represent the theoretically best performance. These methods can be contrasted with regression techniques that seek to explain the average behavior within a sample. Frontier techniques can be divided into two types: parametric and non-parametric. Parametric techniques specify a frontier function to be fitted to the data, with or without accounting for noise in the data. DEA is a non-parametric approach, meaning that no prior functional form is assumed for the frontier, outside of a simple assumption of piecewise linear connections of units on the frontier. The ability to apply DEA without assuming a functional form is a very powerful characteristic. This means that the analysis can proceed without knowing the production function, which is the way inputs are transformed into outputs. Non-parametric approaches can simultaneously handle multiple inputs and outputs, but do not account for noise in the data, treating all deviations from the frontier as inefficiencies (Cummins and Zi 1998).

As an efficiency measurement and evaluation methodology, DEA is particularly useful in cases where sample units, termed decision making units (DMU), use multiple inputs and outputs, and are operating under comparable conditions. DEA primarily measures technical efficiency, i.e. focusing on levels of inputs relative to outputs, as opposed to economic efficiency which would also consider market prices. The use of levels of inputs and outputs is another powerful characteristic of DEA, in that it can incorporate inputs and outputs in the natural units in which they are measured and does not require them to be converted to the same units of measure – specifically, they are not required to be converted to monetary units.

DEA permits the evaluated DMUs to appear to be as good as possible, a feature that can be deemed as providing a "fair" evaluation of the DMUs in the sense that the analysis should limit objections amongst DMUs regarding their evaluations. This characteristic stems from the optimization underpinning of DEA, where DEA assigns the highest efficiency rating to each DMU compared with the set of DMUs being analyzed. It essentially gives the "benefit of the doubt" to each unit. From a management perspective, DEA will be less likely to erroneously identify an efficient unit as inefficient, and while it may not capture all inefficient units, the ones identified as inefficient will have real potential for improvement.

On the contrary, this same "fairness" can permit DMUs to select evaluation criteria that may be deemed as inappropriate or unrealistic. The DEA results, including the evaluation criteria, can be reviewed and adjusted by the user to rerun DEA to include more appropriate criteria. This is one of several ways DEA can be adapted to the specific operating environment of the DMUs. These adaptations make DEA more powerful but also require that the user understand the nature of these added constraints and how that nature affects the way the DEA results are analyzed.

Another advantage of DEA is that it suggests explicit improvement targets for inefficient DMUs, namely the *benchmark* or point on the frontier to which it is being compared in order to measure its efficiency. Furthermore, this frontier point will be defined as the linear combination of one or more actual DMUs that are efficient (i.e. on the efficient frontier). The inefficient DMU is presented with a relevant set of efficient DMUs, called its *reference set* (sometimes referred to as the efficient reference set). The reference set represents the specific efficient DMUs against which the inefficient DMU is judged to be inefficient, and changes to improve the inefficient DMU can be most directly determined by analyzing differences between the inefficient DMU and its reference set. The unit managers thus receive actionable advice that is perceived by them as fair and equitable. Identifying the amount of excess resources consumed or potential increase in outputs possible in inefficient units compared to the DMUs in the efficient reference set may be the most powerful and useful feature of DEA. This perspective offered by DEA is unique, in that it is not provided by any other method known to the authors.

If one were to reread these introductory paragraphs, the clear implication is that DEA is an extremely powerful analytic and management tool. We believe it has been underutilized and hope this volume will open the path to greater utilization. At the same time, we emphasize that DEA is a complement to operating and financial analytical tools, and is not offered as a replacement or a method that must be used exclusively for enhancing business operations.

Model Types

There are three types of basic DEA models: radial, additive and slack-based measure models. These models are detailed in the following sections, along with a discussion of their properties.

Radial Models: CCR

The original DEA model proposed by Charnes et al. (1978), also termed the CCR model, was a radial model. In such a model, a DMU's efficiency score is derived from the extent to which all of its inputs can be contracted and/or its outputs expanded, where this contraction or expansion occurs proportionately. For example, in the case of a model seeking to reduce inputs, the greatest percentage reduction in all inputs is sought; hence the term "radial", as the examined input possibilities occur on the line extending radially from the origin of the input space

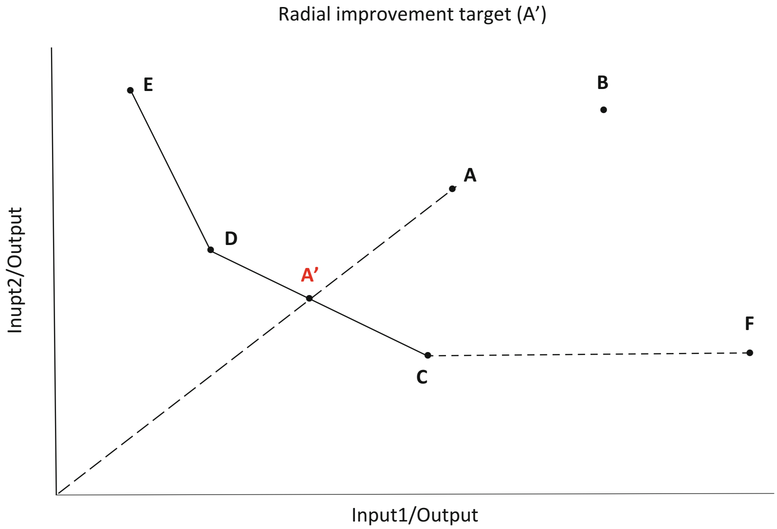

Fig. 1.1 Radial improvement target (A′) from CCR model for a 2-input and 1-output case

(i.e. zero values for all inputs) to the DMU's original inputs (c.f. DMU A in Fig. 1.1).

The CCR model assumed that the production technology, also called the production possibility set, exhibited constant returns to scale (CRS). Charnes, Cooper and Rhodes gave the model in two orientations: input- and output-orientations. The orientation corresponds to the viewpoint taken in improving the inefficient units, whether the goal would be to reduce excess inputs consumed or expand shortfalls in outputs produced, respectively, to move the inefficient unit to the frontier. The frontier – sometimes referred to as the best practice frontier – in Fig. 1.1 is composed of the lines joining points EDC. The line joining CF is termed a *weakly efficient skirt* of the frontier, since points on this line – other than C – will have a radial efficiency score of one, but could still reduce Input 1 without adversely affecting other variables, see Eq. 1.3 below. For a model with m inputs variables, s output variables, and n DMUs, the envelopment form of the input-oriented model is given by Cooper et al. (2007):

$$\begin{aligned} & \min_{\theta,\lambda} \theta \\ \text{subject to} \quad & \theta x_o - \boldsymbol{X}\lambda \geq 0 \\ & \boldsymbol{Y}\lambda \geq y_o \\ & \lambda \geq 0, \end{aligned} \tag{1.1}$$

where x_o and y_o are the column vectors of inputs and outputs respectively for DMU_o, **X** and **Y** are the matrices of input and output vectors respectively for all DMUs, λ is the column vector of intensity variables denoting linear combinations of DMUs, and the objective function θ is a radial contraction factor that can be applied to DMU$_o$'s inputs. As DEA measures efficiency *empirically* relative to the data sample, having too few DMUs will generally result in a large proportion of them being found to be efficient. A general rule of thumb as to the minimum number of DMUs in relation to the number of variables to have a meaningful result with a clear set of efficient and inefficient units is given by Banker et al. (1989):

$$n \geq \max\{m \times s, 3(m+s)\}, \tag{1.2}$$

where m, s and n are the numbers of inputs, outputs and DMUs respectively. This is more a rule of thumb than a rule, which by its nature is a qualitative judgment. When using DEA, it is possible to get very useful results with fewer DMUs than are suggested by this guideline as long as the results are analyzed understanding that there is a small sample of DMUs and thus limited discriminatory power in the model.

The model given in Eq. 1.1 seeks to identify the largest proportion by which all inputs can be reduced (i.e. 1-θ), while at least producing the same level of outputs as the original DMU. Also note that it is assumed that all linear combinations of two or more actual DMUs also represent possible productions, i.e. combinations of inputs and outputs. The linear programming (LP) optimization given in Eq. 1.1 is repeated for each DMU. The optimal value of θ, denoted by θ^*, obtained can be considered the efficiency score of the DMU in question, and this value will range from zero to one, inclusive. (Frequently, the 0–1 scores are reported as percentages – 0% to 100%.) Efficient units will not be able to further reduce inputs and hence have an efficiency score (θ^*) of one. The efficient unit with a rating of 1 or 100% is relatively efficient compared to the DMUs in the study and is not represented as having reached absolute efficiency in an engineering of theoretical sense.

In some instances, it may be possible to further improve the DMU's production performance after the radial optimization. For example, in an input-oriented model, it may be possible to reduce the usage of the first input to 80% of the initial amount, while only reducing the remainder of the inputs to 85%. The θ^* would be 0.85, but in suggesting an improvement target for the DMU, it would be more intuitive to incorporate the additional possible improvement in the first input. Similarly, even though the input-oriented model focuses on reducing inputs, it may be possible to produce more outputs using the same amount of inputs. These additional possible input reductions and output expansions are termed *slacks*, and can be optimized through a second stage to the DEA model, which is given as Eq. 1.3 (Cooper et al. 2007):

$$\begin{aligned} \max_{\lambda, s^-, s^+} \quad & \omega = e_m s^- + e_s s^+ \\ \text{subject to} \quad & s^- = \theta^* x_o - \boldsymbol{X}\lambda \\ & s^+ = \boldsymbol{Y}\lambda - \boldsymbol{y_o} \\ & \lambda \geq 0, s^- \geq 0, s^+ \geq 0, \end{aligned} \tag{1.3}$$

where e_m and e_s are row vectors of m and s ones respectively, s^- and s^+ are column vectors of input and output slacks respectively, θ* is the optimal input contraction obtained from the first stage (Eq. 1.1), and the remaining variables are as previously described.

Equation 1.1 is termed the *envelopment* form of DEA. The same model can be presented in another, equivalent model, termed the *multiplier* form (Cooper et al. 2007):

$$\begin{aligned} \max_{v,u} \quad & uy_o \\ \text{subject to} \quad & vx_o = 1 \\ & -vX + uY \le 0 \\ & u \ge 0, v \ge 0. \end{aligned} \tag{1.4}$$

As with Eq. 1.1, the multiplier form of the DEA model is run once for each DMU in the sample. The model selects virtual or marginal weights for the input and output variables, v's and u's respectively, in such a way as to maximize the efficiency score of the $\mathrm{DMU_o}$, where efficiency is measured as the ratio of the virtual output (i.e. sum of outputs weighted by the virtual weights) to the virtual input. The only restriction on the chosen weights are that they be non-negative and feasible for the sample, i.e. that applying the same weights to any DMU in the sample will not produce an efficiency score greater than one. It is this interpretation of the multiplier form of DEA that lends to the prior assertion of the fairness of DEA models to the evaluated DMUs.

Note that the above model allows the weights, u and v to be greater than or equal to zero. The intention is that the weights should be greater than zero, as allowing a weight to be zero effectively eliminates that input or output from the assessment of a DMU. For computational and other reasons, some DEA programs allow zero weights. Some may use a very small minimum value to at least include all inputs and outputs in the assessment of every DMU in the dataset. If one uses a DEA program and there are zero weights, the interpretation of the results should explicitly consider the implications of the zero weights, as each DMU can look relatively more efficient by removing the inputs/outputs that it tends to use/produce least efficiently via assigning zero weight to those inputs and outputs. Most commercial DEA software will run both envelopment and multiplier forms of the models, as well as any second-stage slack optimizations.

The envelopment and multiplier forms of the model form a primal-dual pair of LPs, and as such the optimal solutions to Eqs. 1.1 and 1.4, and thus the determined efficiency scores, will be the same. Any LP problem, termed the *primal*, can be transformed, through a set procedures known as *taking the dual* (c.f. Appendix A, Cooper et al. 2007) into another LP, the *dual*, and the optimal solutions to each of the two will be the same, provided a solution exists. The primal-dual terminology is non-specific, since taking the dual of the dual program retrieves the original primal program. As such, each of two could be considered the primal or the dual. Hence, this book will avoid the labels *primal* and *dual* and instead employ the more descriptive and specific labels of *envelopment* and *multiplier* forms of DEA. It should be noted that most DEA studies tend to refer to the multiplier form as the primal LP model.

The envelopment form of the output-oriented CRS radial model is presented as Eq. 1.5 (Cooper et al. 2007).

$$\begin{array}{ll} \max\limits_{\phi,\lambda} & \phi \\ \text{subject to} & x_o - \boldsymbol{X}\lambda \geq 0 \\ & \boldsymbol{Y}\lambda \geq \phi y_o \\ & \lambda \geq 0, \end{array} \tag{1.5}$$

where φ is the radial expansion factor that can be applied to DMU_o's outputs.

Analogous to the input-oriented version of the model, the model seeks the maximum factor by which all outputs can be simultaneously expanded. Taking the inverse of the optimal expansion factor, φ^*, produces an efficiency score in the standard sense, i.e. ranging from zero to one. Efficient units will not be able to increase outputs produced from the same inputs, and thus have a φ^* and efficiency score of one. One property of radial CRS DEA models is that the efficiency scores determined for DMUs are the same in both input- and output-oriented models, i.e. $\theta^* = 1/\varphi^*$. For the multiplier form of the output-oriented CCR model, refer to Cooper et al. (2007).

Appendix – Basic DEA model illustration: The DEA model assuming no knowledge of linear programming is explained and applied to a simple dataset in the appendix to this chapter. This fundamental description of DEA illustrates the way it might be used to identify best practice DMUs, inefficient DMUs, and the potential benefits if inefficient DMUs become as efficient as the efficient DMUs by making the changes suggested by DEA.

Radial Models: BCC

Banker et al. (1984) developed a radial DEA model where the production technology exhibits variable returns to scale (VRS). The envelopment form of the input-oriented version of the model is given by Cooper et al. (2007):

$$\begin{array}{ll} \min\limits_{\theta_B,\lambda} & \theta_B \\ \text{subject to} & \theta_B x_o - \boldsymbol{X}\lambda \geq 0 \\ & \boldsymbol{Y}\lambda \geq y_o \\ & e_n\lambda = 1 \\ & \lambda \geq 0. \end{array} \tag{1.6}$$

Comparing Eqs. 1.1 and 1.6, it can be seen that they differ in the addition of a constraint that the sum of the intensity variables, λ's, be equal to one in the VRS model. The effect of this constraint is to limit a DMU to being compared to other

DMUs that are of roughly the same operational scale, which allows for the existence of VRS, i.e. increasing, constant or decreasing returns to scale (RTS). The CRS efficiency score will be less than or equal to the VRS score, and the ratio of CRS/VRS scores gives a measure of the DMU's scale efficiency, i.e. the effect on its productivity from potentially not operating at the optimal scale. This relationship between CRS and VRS scores holds for all DEA models.

Varying the constraint on the sum of λ's to being less than or equal to one results in a non-increasing returns to scale (i.e. permitting constant or decreasing RTS) model. Restricting the sum of lambdas to be greater than or equal to one yields a non-decreasing RTS model. The effect of these constraints on the λ's affects the RTS properties of other DEA models in the same manner.

The VRS model is frequently applied and can offer useful additional insights to those obtained from a CRS model on the same dataset. When the VRS or CRS model is specified, the reason for choosing one over the other should also be noted. One of the advantages of DEA is that one need not know the functional form, which would include knowing the returns to scale characteristics. There are also cases where a larger unit is less efficient than a smaller unit, and analyzing this situation where there are expected to be increasing returns to scale can overlook the real possibility that the large unit is less efficient due to the way it operates and not due to decreasing returns or any scale affect. Applying both CRS and VRS would help identify the inefficiency in the larger unit.

The reader is referred to Cooper et al. (2007) for the multiplier form of the input-oriented BCC model, as well as the two formulations for its output-oriented version. Note that unlike the case for CRS models, it is not generally the case that the efficiency scores from the input- and output-oriented versions of VRS models will be the same.

In some analysis situations, there may not be an intuitive reason to emphasize either input reduction or output maximization, and instead it may be reasonable to pursue both. To address this situation, radial DEA models can express in a non-oriented form. The CRS version of the envelopment form of the non-oriented radial model is (Tam 2004):

$$\begin{array}{ll} \max\limits_{g,\bar{\lambda}} & g \\ \text{subject to} & x_o - \boldsymbol{X}\bar{\lambda} \geq 0 \\ & \boldsymbol{Y}\bar{\lambda} \geq g y_o \\ & g \geq e_n \bar{\lambda} \\ & \bar{\lambda} \geq 0, \end{array} \tag{1.7}$$

where g = φ/θ, $\bar{\lambda} = \lambda/\theta$, and the other variables are as defined in Eqs. 1.1, 1.5, and 1.6. In this model both a radial contraction, θ, and radial expansion, φ, are determined, subject to constraints that the target to which a DMU is being compared cannot use more inputs (θ ≤ 1) or produce less outputs (φ ≥ 1). The efficiency score for the DMU is given by $1/g^* = \theta^*/\varphi^*$. For the CRS model, the efficiency score from the non-oriented radial model will be the same as those obtained from the input- and output-oriented models.

Tam (2004) also gave a non-oriented radial model operating under VRS, presented as (Eq. 1.8):

$$\begin{aligned} \max_{g,\bar{\lambda}} \quad & g \\ \text{subject to} \quad & x_o - \boldsymbol{X}\bar{\lambda} \geq 0 \\ & \boldsymbol{Y}\bar{\lambda} \geq g y_o \\ & e_n\bar{\lambda} \geq 1 \\ & g \geq e_n\bar{\lambda} \\ & \bar{\lambda} \geq 0. \end{aligned} \tag{1.8}$$

In the case of VRS models, the efficiency scores from Eq. 1.8 will be less than or equal to scores for the same DMUs in both the input- and output-oriented VRS radial models, i.e. Eqs. 1.1 and 1.5. The multiplier forms of the non-oriented radial models can be found in Tam (2004).

Additive Models

DEA is most useful for modelling production situations involving multiple inputs and multiple outputs. One of the inherent difficulties in dealing with these situations is the evaluation of trade-offs, for example between substituting one input for another. This evaluation is referred to as considering the mix or allocative efficiency of the DMUs. In situations with known prices for all inputs and outputs, the cost, revenue or profit can be optimized to decide upon the best input and/or output mixes. However, in many situations, prices or values are not known or not fixed for all inputs and outputs. Radial DEA models generally avoid dealing with mix issues by looking at proportional changes to inputs and outputs in their first stage. Proportional changes keep the input and output mixes the same as those originally employed by the DMU.

The additive model of DEA does address the input and output mixes of the DMUs. Its goal is to determine the maximum extent to which slacks can be removed from the DMU being evaluated. It is generally used as a non-oriented model, the VRS envelopment form of which is given as [refer to Cooper et al. (2007) – the multiplier form of the additive model can also be found therein]:

$$\begin{aligned} \max_{\lambda, s^-, s^+} \quad & z = e_m s^- + e_s s^+ \\ \text{subject to} \quad & \boldsymbol{X}\lambda + s^- = x_o \\ & \boldsymbol{Y}\lambda - s^+ = y_o \\ & e_n\lambda = 1 \\ & \lambda \geq 0, s^- \geq 0, s^+ \geq 0. \end{aligned} \tag{1.9}$$

The characteristics of the additive model are very different from those of the radial DEA models. Its results are not easily expressed as standard efficiency scores, i.e. values ranging from zero to one, with one representing efficiency. The optimal objective function value for efficient units in Eq. 1.9 is zero, as efficient units will have no slacks, and there is no defined upper limit on the total slacks. Unlike most forms of DEA (e.g. radial and slack-based measure models), the additive model can have zeros or negative values in the variable data, and is translation invariant, meaning that a constant could be added or subtracted from the values of a particular variable across all the DMUs without affecting the results. However, unlike most other DEA models, it is not unit invariant, and as such measuring a variable in miles as opposed to kilometers could affect the analysis results.

SBM Model

Tone (2001) formulated the slack-based measure (SBM) as a development of the additive model that would generate a standard efficiency score and be unit invariant, while also allowing for input and/or output mix considerations. The envelopment form of the input-oriented CRS SBM is given by:

$$\begin{aligned}
\min_{\lambda, s^-, s^+} \quad & \rho = 1 - \frac{1}{m}\sum_{i=1}^{m} s_i^- / x_{io} \\
\text{subject to} \quad & x_0 = \boldsymbol{X}\lambda + s^- \\
& y_0 = \boldsymbol{Y}\lambda - s^+ \\
& \lambda \geq 0, s^- \geq 0, s^+ \geq 0.
\end{aligned} \tag{1.10}$$

From Eq. 1.10, it can be seen that the SBM, like the additive model, is maximizing the total input slacks, but the slacks are considered as a proportion of the initial input value, as opposed to being considered in absolute terms. Similarly, output- and non-oriented, and VRS forms of the SBM, as well as corresponding multiplier forms of these models can be formulated, c.f. Tone (2001) and Cooper et al. (2007). The input- and output-oriented SBM models could undergo a second stage slack optimization, as occurs with radial DEA models, in the outputs and inputs respectively.

Comparing Eqs. 1.1 and 1.10, it can be seen that the SBM is similar in form and function to a radial DEA model. Whereas an input-oriented radial DEA model maximizes the proportional input contraction that is applied to all input variables, the SBM model maximizes the average proportional input contraction across all the inputs. Hence the SBM model is implicitly assuming that a 1% reduction in one input has the exact same value as 1% reduction in any other input, or as another example, the combination of a 0.4% reduction in a second input and a 0.6% reduction in a third input. Further, it can be noted that the efficiency score from an SBM model will be less than or equal to that from the corresponding (i.e. same orientation and RTS assumption) radial DEA model.