Wolfgang Paul · Jörg Baschnagel

Stochastic Processes

From Physics to Finance

2nd Edition

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Preface to the Second Edition

Thirteen years have passed by since the publication of the first edition of this book. Its favorable reception encouraged us to work on this second edition. We took advantage of this opportunity to update the references and to correct several mistakes which (inevitably) occurred in the first edition. Furthermore we added several new sections in Chaps. 2, 3 and 5. In Chap. 2 we give an introduction to Jaynes' treatment of probability theory as a form of logic employed to judge rational expectations and to his famous maximum entropy principle. Additionally, we now also discuss limiting distributions for extreme values. In Chap. 3 we added a section on the Caldeira-Leggett model which allows to derive a (generalized) Langevin equation starting from the deterministic Newtonian description of the dynamics. Furthermore there is now also a section about the first passage time problem for unbounded diffusion as an example of the power of renewal equation techniques and a discussion of the extreme excursions of Brownian motion. Finally we extended the section on Nelson's stochastic mechanics by giving a detailed discussion on the treatment of the tunnel effect. Chapter 5 of the first edition contained a discussion of credit risk, which was based on the commonly accepted understanding at that time. This discussion has been made obsolete by the upheavals on the financial market occurring since 2008, and we changed it accordingly. We now also address a problem that has been discussed much in the recent literature, the (possible) non-stationarity of financial time series and its consequences. Furthermore, we extended the discussion of microscopic modeling approaches by introducing agent based modeling techniques. These models allow to correlate the behavior of the agents—the microscopic 'degrees of freedom' of the financial market—with the features of the simulated financial time series, thereby providing insight into possible underlying mechanisms. Finally, we also augmented the discussion about the description of extreme events in financial time series.

We would like to extend the acknowledgments of the first edition to thank M. Ebert, T. Preis and T. Schwiertz for fruitful collaborations on the modeling of financial markets. Without their contribution Chap. 5 would not have its present form.

Halle and Strasbourg, February 2013

Wolfgang Paul
Jörg Baschnagel

Preface to the First Edition

Twice a week, the condensed matter theory group of the University of Mainz meets for a coffee break. During these breaks we usually exchange ideas about physics, discuss administrative duties, or try to tackle problems with hardware and software maintenance. All of this seems quite natural for physicists doing computer simulations.

However, about two years ago a new topic arose in these routine conversations. There were some Ph.D. students who started discussions about the financial market. They had founded a 'working group on finance and physics'. The group met frequently to study textbooks on 'derivatives', such as 'options' and 'futures', and to work through recent articles from the emerging 'econophysics' community which is trying to apply well-established physical concepts to the financial market. Furthermore, the students organized special seminars on these subjects. They invited speakers from banks and consultancy firms who work in the field of 'risk management'. Although these seminars took place in the late afternoon and sometimes had to be postponed at short notice, they were better attended than some of our regular group seminars. This lively interest evidently arose partly for the reason that banks, insurance companies, and consultancy firms currently hire many physicists. Sure enough, the members of the 'working group' found jobs in this field after graduating.

It was this initiative and the professional success of our students that encouraged us to expand our course on 'Stochastic Processes' to include a part dealing with applications to finance. The course, held in the winter semester 1998/1999, was well attended and lively discussions throughout all parts gave us much enjoyment. This book has accrued from these lectures. It is meant as a textbook for graduate students who want to learn about concepts and 'tricks of the trade' of stochastic processes and to get an introduction to the modeling of financial markets and financial derivatives based on the theory of stochastic processes. It is mainly oriented towards students with a physics or chemistry background as far as our decisions about what constituted 'simple' and 'illustrative' examples are concerned. Nevertheless, we tried to keep our exposition so self-contained that it is hopefully also interesting and helpful to students with a background in mathematics, economics or engineering. The book is also meant as a guide for our colleagues who may plan to teach a similar course.

The selection of applications is a very personal one and by no means exhaustive. Our intention was to combine classical subjects, such as random walks and Brownian motion, with non-conventional themes.

One example of the latter is the financial part and the treatment of 'geometric Brownian motion'. Geometric Brownian motion is a viable model for the time evolution of stock prices. It underlies the Black-Scholes theory for option pricing, which was honored by the Nobel Prize for economics in 1997.

An example from physics is Nelson's 'stochastic mechanics'. In 1966, Nelson presented a derivation of non-relativistic quantum mechanics based on Brownian motion. The relevant stochastic processes are energy-conserving diffusion processes. The consequences of this approach still constitute a field of active research.

A final example comprises stable distributions. In the 1930s the mathematicians Lévy and Khintchine searched for all possible limiting distributions which could occur for sums of random variables. They discovered that these distributions have to be 'stable', and formulated a generalization of the central limit theorem. Whereas the central limit theorem is intimately related to Brownian diffusive motion, stable distributions offer a natural approach to anomalous diffusion, i.e., subdiffusive or superdiffusive behavior. Lévy's and Khintchine's works are therefore not only of mathematical interest; progressively they find applications in physics, chemistry, biology and the financial market.

All of these examples should show that the field of stochastic processes is copious and attractive, with applications in fields as diverse as physics and finance. The theory of stochastic processes is the 'golden thread' which provides the connection. Since our choice of examples is naturally incomplete, we have added at the end of each chapter references to the pertinent literature from which we have greatly profited, and which we believe to be excellent sources for further information. We have chosen a mixed style of referencing. The reference section at the end of the book is in alphabetical order to group the work of a given author and facilitate its location. To reduce interruption of the text we cite these references, however, by number.

In total, the book consists of five chapters and six appendices, which are structured as follows. Chapter 1 serves as an introduction. It briefly sketches the history of probability theory. An important issue in this development was the problem of the random walk. The solution of this problem in one dimension is given in detail in the second part of the chapter. With this, we aim to provide an easy stepping stone onto the concepts and techniques typical in the treatment of stochastic processes.

Chapter 2 formalizes many of the ideas of the previous chapter in a mathematical language. The first part of the chapter begins with the measure theoretic formalization of probabilities, but quickly specializes to the presentation in terms of probability densities over $\mathbb{R}^d$. This presentation will then be used throughout the remainder of the book. The abstract definitions may be skipped on first reading, but are included to provide a key to the mathematical literature on stochastic processes. The second part of the chapter introduces several levels of description of Markov processes (stochastic processes without memory) and their interrelations, starting from the Chapman-Kolmogorov equation. All the ensuing applications will be Markov processes.

Chapter 3 revisits Brownian motion. The first three sections cover classical applications of the theory of stochastic processes. The chapter begins with random walks on a d-dimensional lattice. It derives the probability that a random walker will be at a lattice point $\boldsymbol{r}$ after N steps, and thereby answers 'Polya's question': What is the probability of return to the origin on a d-dimensional lattice? The second section discusses the original Brownian motion problem, i.e., the irregular motion of a heavy particle immersed in a fluid of lighter particles. The same type of motion can occur in an external potential which acts as a barrier to the motion. When asking about the time it takes the particle to overcome that barrier, we are treating the so-called 'Kramers problem'. The solution of this problem is given in the third section of the chapter. The fourth section treats the mean field approximation of the Ising model. It is chosen as a vehicle to present a discussion of the static (probabilistic) structure as well as the kinetic (stochastic) behavior of a model, using the various levels of description of Markov processes introduced in Chap. 2. The chapter concludes with Nelson's stochastic mechanics to show that diffusion processes are not necessarily dissipative (consume energy), but can conserve energy. We will see that one such process is non-relativistic quantum mechanics.

Chapter 4 leaves the realm of Brownian motion and of the central limit theorem. It introduces stable distributions and Lévy processes. The chapter starts with some mathematical background on stable distributions. The distinguishing feature of these distributions is the presence of long-ranged power-law tails, which might lead to the divergence of even the lowest-order moments. Physically speaking, these lower-order moments set the pertinent time and length scales. For instance, they define the diffusion coefficient in the case of Brownian motion. The divergence of these moments therefore implies deviations from normal diffusion. We present two examples, one for superdiffusive behavior and one for subdiffusive behavior. The chapter closes with a special variant of a Lévy process, the truncated Lévy flight, which has been proposed as a possible description of the time evolution of stock prices.

The final chapter (Chap. 5) deals with the modeling of financial markets. It differs from the previous chapters in two respects. First, it begins with a fairly verbose introduction to the field. Since we assume our readers are not well acquainted with the notions pertaining to financial markets, we try to compile and explain the terminology and basic ideas carefully. An important model for the time evolution of asset prices is geometric Brownian motion. Built upon it is the Black-Scholes theory for option pricing. As these are standard concepts of the financial market, we discuss them in detail. The second difference to previous chapters is that the last two sections have more of a review character. They do not present well-established knowledge, but rather current opinions which are at the moment strongly advocated by the physics community. Our presentation focuses on those suggestions that employ methods from the theory of stochastic processes. Even within this limited scope, we do not discuss all approaches, but present only a selection of those topics which we believe to fit well in the context of the previous chapters and which are extensively discussed in the current literature. Among those topics are the statistical analysis of financial data and the modeling of crashes.

Finally, some more technical algebra has been relegated to the appendices and we have tried to provide a comprehensive subject index to the book to enable the reader to quickly locate topics of interest.

One incentive for opening or even studying this book could be the hope that it holds the secret to becoming rich. We regret that this is (probably) an illusion. One does not necessarily learn how to make money by reading this book, at least not if this means how to privately trade successfully in financial assets or derivatives. This would require one to own a personal budget which is amenable to statistical treatment, which is true neither for the authors nor probably for most of their readers. However, although it does not provide the 'ABC' to becoming a wizard investor, reading this book can still help to make a living. One may acquire useful knowledge for a prospective professional career in financial risk management. Given the complexity of the current financial market, which will certainly still grow in the future, it is important to understand at least the basic parts of it.

This will also help to manage the largest risk of them all which has been expressed in a slogan-like fashion by the most successful investor of all time, Warren Buffet [23]:

Risk is not knowing what you're doing.

In order to know what one is doing, a thorough background in economics, a lot of experience, but also a familiarity with the stochastic modeling of the market are important. This book tries to help in the last respect.

To our dismay, we have to admit that we cannot recall all occasions when we obtained advice from students, colleagues and friends. Among others, we are indebted to C. Bennemann, K. Binder, H.-P. Deutsch, H. Frisch, S. Krouchev, A. Schäcker and A. Werner. The last chapter on the financial market would not have its present form without the permission to reproduce artwork from the research of J.-P. Bouchaud, M. Potters and coworkers, of R.N. Mantegna, H.E. Stanley and coworkers, and of A. Johansen, D. Sornette and coworkers. We are very grateful that they kindly and quickly provided the figures requested. Everybody who has worked on a book project knows that the current standards of publishing can hardly be met without the professional support of an experienced publisher like Springer. We have obtained invaluable help from C. Ascheron, A. Lahee and many (unknown) others. Thank you very much.

Mainz, Wolfgang Paul
October 1999 Jörg Baschnagel

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Chapter 1
A First Glimpse of Stochastic Processes

In this introductory chapter we will give a short overview of the history of *probability theory* and *stochastic processes*, and then we will discuss the properties of a simple example of a stochastic process, namely the *random walk* in one dimension. This example will introduce us to many of the typical questions that arise in situations involving *randomness* and to the tools for tackling them, which we will formalize and expand on in subsequent chapters.

1.1 Some History

Let us start this historical introduction with a quote from the superb review article *On the Wonderful World of Random Walks* by E.W. Montroll and M.F. Shlesinger [145], which also contains a more detailed historical account of the development of probability theory:

> *Since traveling was onerous (and expensive), and eating, hunting and wenching generally did not fill the 17th century gentleman's day, two possibilities remained to occupy the empty hours, praying and gambling; many preferred the latter.*

In fact, it is in the area of gambling that the theory of probability and stochastic processes has its origin. People had always engaged in gambling, but it was only through the thinking of the Enlightenment that the outcome of a gambling game was no longer seen as a divine decision, but became amenable to rational thinking and speculation. One of these 17th century gentlemen, a certain Chevalier de Méré, is reported to have posed a question concerning the odds at a gambling game to Pascal (1623–1662). The ensuing exchange of letters between Pascal and Fermat (1601–1665) on this problem is generally seen as the starting point of probability theory.

The first book on probability theory was written by Christiaan Huygens (1629–1695) in 1657 and had the title *De Ratiociniis in Ludo Aleae* (On Reasoning in the Game of Dice). The first mathematical treatise on probability theory in the modern sense was Jakob Bernoulli's (1662–1705) book *Ars Conjectandi* (The Art of Conjecturing), which was published posthumously in 1713. It contained

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