THINKING ABOUT EQUATIONS

A Practical Guide for Developing Mathematical Intuition in the Physical Sciences and Engineering

MATT A. BERNSTEIN, PhD
WILLIAM A. FRIEDMAN, PhD

WILEY

A JOHN WILEY & SONS, INC., PUBLICATION

THINKING ABOUT EQUATIONS

THINKING ABOUT EQUATIONS

A Practical Guide for Developing Mathematical Intuition in the Physical Sciences and Engineering

MATT A. BERNSTEIN, PhD
WILLIAM A. FRIEDMAN, PhD

WILEY

A JOHN WILEY & SONS, INC., PUBLICATION

Published by John Wiley & Sons, Inc., Hoboken, New Jersey.
Published simultaneously in Canada.

Library of Congress Cataloging-in-Publication Data:

Bernstein, Matt A., 1958–
Thinking about equations : a practical guide for developing mathematical intuition in the physical sciences and engineering / Matt A. Bernstein, William A. Friedman.
p. cm.
Includes bibliographical references and index.
ISBN 978-0-470-18620-6 (pbk.)
1. Mathematical analysis. 2. Physical sciences–Mathematics. 3. Engineering–Mathematics. I. Friedman, William A. (William Albert) II. Title.
QA300.B496 2009
515—dc22

2009001787

Printed in the United States of America.

10 9 8 7 6 5 4 3 2 1

CONTENTS

PREFACE

Equations play a central role in day-to-day problem solving for the physical sciences, engineering, and related fields. Those of us who pursue these technical careers must learn to understand and to solve equations. It is often a challenge to figure out what an equation means and then to form a strategy to solve it. One aim of this book is to illustrate that by investing a little effort into thinking about equations, your task can become easier and your solutions can become more reliable.

On a deeper level, studying the topics presented here could provide you with greater insight into your own field of study, and you may even derive some intellectual pleasure in the process. Today, a staggering amount of knowledge is readily available through the Internet. With this abundance of information comes an ever-greater premium on understanding the interconnections and themes embedded within this information. This book can help you gain that intuitive understanding while you acquire some practical mathematical tools. We hope that our selection of topics and approaches will open your eyes to this useful and rewarding process.

One might ask why we wrote a book about equations, i.e., focusing on analytic methods, when today so many problems are solved using numerical methods. In our own scientific research, we rely heavily on numerical computation. Equations, however, are the basis for almost all numerical solutions and simulations. A deeper understanding of the underlying equations often can improve the strategy for a numerical approach, and importantly, it helps one to check whether the resulting numerical answers are reasonable.

With the advent and widespread availability of powerful computational packages such as MATLAB®, MAPLE™, IDL®, and others, some students have spent less time studying and thinking about equations and have not yet developed an intuitive "feel" for this topic that experience brings. This book can serve as a resource and workbook for students who wish to improve those skills in an efficient manner. We believe that the powerful graphics capabilities of those computational packages can complement the material presented here, and users of symbolic manipulation software such as Mathematica® will also find this book relevant to their work.

The core of our intended audience is undergraduate students majoring in any discipline within engineering or any of the physical sciences including physics, geology, astronomy, and chemistry, as well as related fields such as applied mathematics and biophysics. Even students in peripherally related fields, such as computational finance, will find the book useful, particularly when they apply methods that originated in the physical sciences and engineering to their own work. Graduate students with an undergraduate background in a field like biology who are transitioning into a more quantitative area such as biomedical engineering could also find the techniques described in this book quite useful.

We have tried to target the level of the book so it is accessible to undergraduate students who have a background in basic calculus and some familiarity with differential equations. An undergraduate course in introductory physics is also recommended. Some of the worked example problems draw on concepts and results from linear algebra and complex analysis, but this background is not essential to follow most of the material. We are not mathematicians, and there are no formal mathematical proofs in the book. Instead, the derivations, where provided, are presented at a level of rigor that is typical of work in the physical sciences. A few of the sections and example problems are labeled with an asterisk (*), because either they assume a higher level of mathematical background or else the material contained in them is not central to the rest of the book. These starred sections and problems can be skipped without loss of continuity.

This book is *not* a comprehensive "how-to" manual of mathematical methods. Instead, selected concepts are introduced by a short discussion in each section of each chapter. Then, the concepts are illustrated and further developed with example problems followed by detailed solutions. These worked example problems form the backbone of this book.

Our own backgrounds are in physics, with specialization in magnetic resonance imaging, medical physics, and biomedical engineering (MAB) and theoretical nuclear physics (WAF). While we have chosen example problems drawn from subjects that are familiar to us, we have tried to avoid problems that require highly specialized knowledge. We also have tried to provide explanations of terms that might be unfamiliar and to avoid the use of jargon. We hope you will read and work through example problems that are *outside* of your immediate area of interest. If you can recognize concepts that can be applied to your own work, then you will have made great progress toward our ultimate goal. At the end of each chapter, there are a number of exercises designed to test and further develop your understanding of the material. We also encourage you to devise and to solve some of your own exercises, perhaps related to your own specific field of interest.

Chapter 1 deals with units and dimensions of physical quantities. Although this is a very basic topic, it has some surprisingly profound implications that are developed more fully in Chapter 6, which deals with dimensional analysis and scaling. Chapter 2 illustrates several common pitfalls to avoid when dealing with equations, along with a few handy techniques and tricks that are used in later chapters. Chapter 3 discusses special cases and their use to check equations and to guess at the solution of difficult problems. Chapter 4 focuses on pictorial and graphical methods, and provides a basic introduction to the concept of symmetry and its use for simplifying equations. Chapter 5 discusses a variety of estimation and approximation techniques. Chapter 7 discusses how and when to generalize equations. Finally, Chapter 8 provides a few more instructive examples that illustrate several problem-solving techniques while reinforcing some of the concepts introduced in earlier chapters.

There are many interrelations and cross-references among the chapters. Generalization, which is discussed in Chapter 7, is closely related to the subject of special cases introduced in Chapter 3. Similarly, dimensional analysis and scaling (Chapter 6) are closely related to the study of units and dimensions (Chapter 1). Notice that these linked chapters are spaced apart in the book. This placement is intended to give the reader time to think about the material before it is revisited. Despite the interconnections among them, the chapters are, for the most part, self-contained and need not necessarily be read in numerical sequence. The book can be used to construct modules for a quick study of a more focused topic. Two such possible modules are:

(1) dimensional analysis and scaling: Chapters 1 and 6;
(2) special cases, approximation, and generalization: Chapters 3, 5, and 7.

We have also tried to enliven the discussion by identifying the nationality and approximate era of some of the luminaries who made major contributions to the scientific and mathematical literature that we have drawn upon, and by providing a few historical anecdotes. While this material is not necessary for dealing with the equations themselves, we hope that you find it entertaining and that it will provide some insight into how science often has progressed along an unexpected or a roundabout path.

If you have suggestions for additional material that could be included in future editions of this book, find errors or descriptions that are unclear, or wish to provide feedback of any other type, we would be happy to hear from you.

Matt A. Bernstein
Rochester, Minnesota
mbernstein@mayo.edu
William A. Friedman
Seattle, Washington
friedman@physics.wisc.edu

December 2008

ACKNOWLEDGEMENTS

Several individuals offered their time and effort to review the manuscript and provide comments. We especially want to thank Juliet Bernstein, who offered many insightful suggestions, which have greatly improved the clarity of the book. Kristi Welle reviewed several of the example problems, and Linda Greene proofread many of the pages. We thank Professor Željko Bajzer for showing us the curious integral that is discussed in Example 2.1. We thank them all for their outstanding efforts and emphasize that any remaining mistakes and inaccuracies are solely our own.

Individually, we would like to express our gratitude.

MAB: I thank my friends and coworkers at the Mayo Clinic and also in the broader magnetic resonance imaging community. It has been a privilege to work with such outstanding people during such an exciting time. I am also indebted to my entire family and especially to my wife Rhoda Lichy for her unwavering support and patience during the writing of this book.

WAF: I am grateful to the many students at all levels with whom I have had contact during my nearly 40 years of teaching, and to my late wife Cheryl who was by my side during that journey.

LIST OF WORKED-OUT EXAMPLE PROBLEMS