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Mathematical Finance

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Mathematical Finance

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To Carl-Philipp, Friederike, Sophie and to Birgit, Dörte, Matthies

Preface

Mathematical finance provides a quantitative description of financial markets, more specifically markets for exchange-traded assets, using mostly dynamic stochastic models. It is used to tackle three basic issues.

- *Valuation of assets*
  What can reasonably be said about the price of a financial asset? As opposed to economic theory, mathematical finance focuses mainly on *relative* valuation of securities in comparison to other assets. This is particularly useful and in fact indispensable for derivative securities, which are by definition strongly linked to corresponding underlying quantities in the market.
- *Optimal or at least reasonable portfolio selection*
  How shall an investor choose her portfolio of liquid securities? Here, the focus is on *hedging*, i.e. on minimising the risk which arises, for example, from selling derivative contracts to customers.
- *Quantification of risk*
  The random nature of asset prices naturally involves the risk of losses. How can it be quantified reasonably?

Mathematical finance has grown into a field which is by far too broad to be covered in a single book. Markets, products and risks are diverse and so are the mathematical models and methods which they require.

The starting point and focal point of this present monograph is continuous-time stochastic processes allowing for jumps. Most textbooks on mathematical finance are limited to diffusion-type setups, which cannot easily account for abrupt price movements. Such changes, however, play an important role in real markets, which is why models with jumps have become an established tool in the statistics and mathematics of finance. Just as importantly, purely discontinuous processes lead to a much wider variety of, at the same time flexible and tractable, models. For example, their marginal laws are often known explicitly, which is typically not the case for diffusions.

Compared to the abundant literature on continuous models, such as [29, 78, 149, 187, 204, 223, 223, 279], and many more, there still seems to be a scarcity of textbooks allowing for processes with jumps. Notable exceptions are the monographs [60] and [38, 160, 276]. Other useful texts such as [143, 263], address more specific questions rather than general principles of financial mathematics.

Our goal is twofold:

- to give an account of general semimartingale theory, stochastic control and specific classes of processes to the extent needed for the applications in the second part
- to introduce basic concepts such as arbitrage theory, hedging, valuation principles, portfolio choice and term structure modelling

In a single monograph, we cannot give a comprehensive overview of stochastic models with and without jumps in mathematical finance. Rather, we provide an introduction to the basic building blocks and principles, helping the reader to understand the advanced research literature and to come up with concrete models and solutions in more specific situations.

The book is divided into two parts. Part I introduces the stochastic analysis of general semimartingales along with the basics of stochastic control theory. We do not cover the whole theory with complete proofs, which can be found in a number of excellent mathematical monographs. Rather, we focus on concepts and results that are needed to apply the theory to questions in mathematical finance. Proofs are mostly replaced by informal illustrations along with references to the literature. Nevertheless, we made an effort to provide mathematically rigorous definitions and theorems.

Part II turns to both advanced models and basic principles of mathematical finance. It differs in style from Part I in the sense that results are stated as engineering-style *rules* rather than precise mathematical theorems with all the technical assumptions. For example, we do not distinguish between local and true martingales, and questions of existence and uniqueness are swept under the rug. This is done deliberately in order to make basic concepts accessible to the mathematically less inclined reader who wants to apply advanced stochastic models in practice and also to the non-specialist who wants to get an overview of the general ideas before delving more thoroughly into the subject.

The theory of Parts I and II simplifies occasionally if one focuses on stochastic processes without jumps. Major changes are summarised in Sect. A.7 for the convenience of the reader. Mathematical finance in the broad sense has produced some insights, which may seem counterintuitive and hence surprising to the novice in the field. We collect links to such results in Sect. A.8. Otherwise, the appendix mostly contains mathematical tools that are needed in the main part on the text.

This book could not have appeared in the present form without the help of many people. An incomplete list includes Aleš Černý, Sören Christensen, Friedrich Hubalek, Simon Kolb, Paul Krühner, Matthias Lenga, Johannes Muhle-Karbe, Arnd Pauwels, and Richard Vierthauer with whom we had long discussions, which had an effect on the contents of the book. Funda Akyüz and Britta Ramthun-Kasan assisted with preparing the manuscript. Partial financial support through *DFG Sachbeihilfe 1682/4-1* is gratefully acknowledged. We also benefited from the environment provided by the Freiburg Institute for Advanced Studies (FRIAS). We thank Catriona Byrne and Marina Reizakis from Springer-Verlag for their interest, encouragement and patience.

Errors can hardly be avoided in a text of this size. Since they will be discovered only gradually, we refer to www.math.uni-kiel.de/finmath/~book for an updated list of corrections. On this page, you can also find the *Scilab* code that we have used to generate the figures and numerical examples. Of course, any comments and in particular hints to errors are welcome.

Freiburg im Breisgau, Germany — Ernst Eberlein
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Part I
Stochastic Calculus

Overview

A large part of mathematics for finance is written in the language of *stochastic processes*, i.e. of random functions of time. The calculus of these processes is introduced in this first part of the monograph. Some of its concepts naturally generalise notions from ordinary calculus, others are intrinsically linked to their probabilistic nature.

Even though this book deals with continuous-time models, we devote the first chapter to stochastic calculus in discrete time. The results themselves will not be needed in the sequel but they help to understand the intuition behind the corresponding concepts in continuous time.

The remaining chapters generalise important notions from ordinary calculus to the random case. *Lévy processes* can be viewed as the stochastic counterparts of linear functions. They are of interest in their own right but they also appear as building blocks of more general classes of processes. In Chap. 3 we cover the theory of *stochastic integration*, which is indispensable for mathematical finance. By contrast, it seems less obvious whether and how differentiation can be transferred to the random case. In Chaps. 4 and 5 we discuss *semimartingale characteristics* and *infinitesimal generators* as two natural candidates for a stochastic "differentiation". If Lévy processes and semimartingale characteristics generalise linear functions and derivatives, respectively, *affine Markov processes* correspond to solutions of linear ordinary differential equations. They play an important role in finance because of their flexibility and analytical tractability. Finally, we introduce the basic concepts of *stochastic optimal control* in Chap. 7 because many questions in Mathematical Finance are explicitly or implicitly related to optimisation.

Informal differential notation and arguments are used occasionally in the physics literature and to some extent in finance as well. We mimic such reasoning here in a few so-called *physicist's corners*. While these remarks may be insightful to some readers, they could confuse others with a more formal mathematical background. In the latter case they can be skipped altogether because they are primarily meant to illustrate separately stated rigorous mathematical statements.

Chapter 1
Discrete Stochastic Calculus

The theory of stochastic processes deals with random functions of time such as asset prices, interest rates, and trading strategies. As is also the case for Mathematical Finance, it can be developed in both discrete and continuous time. Actual calculations are often easier and more transparent in continuous-time models, which is why we focus on the latter in this book. However, there is a price to be paid. A completely sound treatment of the continuous case requires considerably more complex mathematical arguments, which are beyond the scope of this monograph. On the other hand, the phenomena and formulae in discrete and continuous time resemble each other quite closely. Therefore we use the simpler discrete case as a means to motivate the technically more demanding results in the subsequent chapters.

1.1 Processes, Stopping Times, Martingales

The natural starting point in probability theory is provided by a *probability space* $(\Omega, \mathscr{F}, P)$. The more or less abstract *sample space* Ω stands for the possible outcomes of a random experiment. For example, it could contain all conceivable sample paths of a stock price process. The *probability measure* P states probabilities of sets of outcomes. For measure-theoretic reasons it is typically impossible to assign probabilities to *all* subsets of Ω in a consistent manner. As a way out one specifies a *σ-field* $\mathscr{F}$, i.e. a collection of subsets of Ω which is closed under countable set operations $\cap$, $\cup$, $\setminus$, ^{C}. If the probability $P(F)$ is defined only for *events* $F \in \mathscr{F}$, one can avoid the paradoxes involved in considering arbitrary sets.

Random variables X are functions of the outcome $\omega \in \Omega$. Typically its values $X(\omega)$ are numbers but they may also be vectors or even functions, in which case X is a *random vector* resp. *process*. We denote by $E(X)$ and $\mathrm{Var}(X)$ the expected value

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and variance, respectively, of a real-valued random variable. Accordingly, $E(X)$ and $\mathrm{Cov}(X)$ denote the expectation vector and covariance matrix, respectively, of a random vector X.

For static random experiments one needs to consider only two states of information. *Before* the experiment nothing precise is known about the outcome, only probabilities and expected values can be assigned. *After* the experiment the outcome is completely determined. In dynamic random experiments such as stock markets the situation is more involved. In the process of observation, some random events (e.g. yesterday's stock returns) have already happened and can be considered as deterministic whereas others (e.g. tomorrow's stock returns) still belong to the unknown future. As time passes, more and more information is accumulated.

This increasing knowledge is expressed mathematically in terms of a **filtration** $\mathbf{F} = (\mathscr{F}_t)_{t\geq 0}$, i.e. an increasing sequence of sub-σ-fields of $\mathscr{F}$. The collection of events $\mathscr{F}_t$ stands for the observable information up to time t. Specifically, the statement $F \in \mathscr{F}_t$ means that the random event F (e.g. $F = \{$stock return positive at time $t-1\}$) is no longer random at time t. We know for sure whether it is true or not. For example, if our observable information is given by the evolution of the stock price, then $\mathscr{F}_t$ contains all events that can be expressed in terms of the stock price up to time t. The quadruple $(\Omega, \mathscr{F}, \mathbf{F}, P)$ is called a **filtered probability space**. We consider it to be fixed during most of the following. Often one assumes $\mathscr{F}_0 = \{\varnothing, \Omega\}$, i.e. $\mathscr{F}_0$ is the *trivial σ-field* corresponding to no prior information.

As time passes, not only the observable information but also probabilities and expectations of future events change. For example, our conception of the terminal stock price evolves gradually from vague ideas to perfect knowledge. This is modelled mathematically in terms of conditional expectations. The *conditional expectation* $E(X|\mathscr{F}_t)$ of a random variable X is its expected value given the information up to time t. As such, it is not a number but itself a random variable which may depend on the randomness up to time t, e.g. on the stock price up to t in the above example. Mathematically speaking, $Y = E(X|\mathscr{F}_t)$ is *$\mathscr{F}_t$-measurable*, which means that $\{Y \in B\} := \{\omega \in \Omega : Y(\omega) \in B\} \in \mathscr{F}_t$ for any reasonable (mathematically phrased: *Borel*) set B. Accordingly, the *conditional probability* $P(F|\mathscr{F}_t)$ denotes the probability of an event $F \in \mathscr{F}$ given the information up to time t. As is true for conditional expectation, it is not a number but an $\mathscr{F}_t$-measurable random variable.

Formally, the **conditional expectation** $E(X|\mathscr{F}_t)$ is defined as the unique $\mathscr{F}_t$-measurable random variable Y such that $E(XZ) = E(YZ)$ for any bounded, $\mathscr{F}_t$-measurable random variable Z. It can also be interpreted as the best prediction of X given $\mathscr{F}_t$. Indeed, if $E(X^2) < \infty$, then $E(X|\mathscr{F}_t)$ minimises the mean squared difference $E((X-Z)^2)$ among all $\mathscr{F}_t$-measurable random variables Z. Strictly speaking, $E(X|\mathscr{F}_t)$ is unique only up to a set of probability 0, i.e. any two *versions* $Y, \widetilde{Y}$ satisfy $P(Y \neq \widetilde{Y}) = 0$. In this book we do not make such fine distinctions. Equations, inequalities etc. are always meant to hold only almost surely, i.e. up to a set of probability 0.

A few rules on conditional expectations are used over and over again. For instance, we have

$$E(X|\mathscr{F}_t) = E(X) \tag{1.1}$$

if $\mathscr{F}_t = \{\varnothing, \Omega\}$ is the trivial σ-field representing no information on random events. More generally, (1.1) holds if X and $\mathscr{F}_t$ are *stochastically independent*, i.e. if

$$P(\{X \in B\} \cap F) = P(X \in B)P(F)$$

for any Borel set B and any $F \in \mathscr{F}_t$. On the other hand we have $E(X|\mathscr{F}_t) = X$ and more generally

$$E(XY|\mathscr{F}_t) = XE(Y|\mathscr{F}_t)$$

if X is $\mathscr{F}_t$-measurable, i.e. known at time t. The **law of iterated expectations** tells us that

$$E\big(E(X|\mathscr{F}_t)\big|\mathscr{F}_s\big) = E(X|\mathscr{F}_s)$$

for $s \leq t$. Almost as a corollary we have

$$E\big(E(X|\mathscr{F}_t)\big) = E(X).$$

Finally, the conditional expectation shares many properties of the expectation, e.g. it is linear and monotone in X and it satisfies monotone and dominated convergence, Fatou's lemma, Jensen's inequality, etc.

Recall that the probability of a set can be expressed as the expectation of an indicator function via $P(F) = E(1_F)$. This suggests to use the relation

$$P(F|\mathscr{F}_t) := E(1_F|\mathscr{F}_t) \tag{1.2}$$

to define conditional probabilities in terms of conditional expectation. Of course, we would like $P(F|\mathscr{F}_t)$ to be a probability measure when it is considered as a function of F. This property, however, is not as evident as it seems because of the null sets involved in the definition of conditional expectation. We do not worry about technical details here and assume instead that we are given a **regular conditional probability**, i.e. a version of $P(F|\mathscr{F}_t)(\omega)$ which, for any fixed ω, is a probability measure when viewed as a function of F. Such a regular version exists in all instances where it is used in this book.

In line with (1.2) we denote by

$$P^{X|\mathscr{F}_t}(B) := P(X \in B|\mathscr{F}_t) := E(1_B(X)|\mathscr{F}_t)$$

the **conditional law** of X given $\mathscr{F}_t$. A useful rule states that

$$E(f(X,Y)|\mathscr{F}_t) = \int f(x,Y)P^{X|\mathscr{F}_t}(dx) \tag{1.3}$$

for real-valued measurable functions f and $\mathscr{F}_t$-measurable random variables Y. If X is stochastically independent of $\mathscr{F}_t$, we have $P^{X|\mathscr{F}_t} = P^X$, i.e. the conditional law of X coincides with the law of X. In this case, (1.3) turns into

$$E(f(X,Y)|\mathscr{F}_t) = \int f(x,Y)P^X(dx) \tag{1.4}$$

for $\mathscr{F}_t$-measurable random variables Y.

A **stochastic process** $X = (X(t))_{t\geq 0}$ is a collection of random variables $X(t)$, indexed by time t. In this chapter the time set is assumed to be $\mathbb{N} = \{0, 1, 2, \dots\}$, afterwards we consider continuous time $\mathbb{R}_+ = [0, \infty)$. As noted earlier, a stochastic process $X = (X(t))_{t\geq 0}$ can be interpreted as a random function of time. Indeed, $X(\omega, t)$ is a function of t (or sequence in the current discrete case) for fixed ω. Sometimes, it is also convenient to interpret a process X as a real-valued function on the product space $\Omega \times \mathbb{N}$ or $\Omega \times \mathbb{R}_+$, respectively. In the discrete time case we use the notation

$$\Delta X(t) := X(t) - X(t-1).$$

Moreover, we denote by $X_- = (X_-(t))_{t\geq 0}$ the process

$$X_-(t) := \begin{cases} X(t-1) & \text{for } t \geq 1, \\ X(0) & \text{for } t = 0. \end{cases}$$

We will only consider processes which are consistent with the information structure, i.e. $X(t)$ is supposed to be observable at time t. Mathematically speaking, we assume $X(t)$ to be $\mathscr{F}_t$-measurable for any t. Such processes X are called **adapted** to the filtration $\mathbf{F}$.

There is in fact a minimal filtration $\mathbf{F}$ such that X is $\mathbf{F}$-adapted. Formally, this filtration is given by

$$\mathscr{F}_t = \sigma(X(s) : s \leq t), \tag{1.5}$$

i.e. $\mathscr{F}_t$ is the smallest σ-field such that all $X(s)$, $s \leq t$, are $\mathscr{F}_t$-measurable. Intuitively, this means that the only available information on random events is coming from observing the process X. One calls $\mathbf{F}$ the **filtration generated by** X.

For some processes one actually needs a stronger notion of measurability than adaptedness, namely *predictability*. A stochastic process X is called **predictable** if $X(t)$ is known already one period in advance, i.e. $X(t)$ is $\mathscr{F}_{t-1}$-measurable. The use of this notion will become clearer in Sect. 1.2.

Example 1.1 (Random Walks and Geometric Random Walks) We call an adapted process X with $X(0) = 0$ a **random walk** (relative to $\mathbf{F}$) if the increments $\Delta X(t), t \geq 1$, are identically distributed and independent of $\mathscr{F}_{t-1}$. We obtain such a process if $\Delta X(t), t \geq 1$ are independent and identically distributed (i.i.d.) random variables and if the filtration $\mathbf{F}$ is generated by X.

Similarly, we call a positive adapted process X with $X(0) = 1$ a **geometric random walk** (relative to $\mathbf{F}$) if the *relative increments*

$$\frac{\Delta X(t)}{X(t-1)} = \frac{X(t)}{X(t-1)} - 1 \tag{1.6}$$

are identically distributed and independent of $\mathscr{F}_{t-1}$ for $t \geq 1$. A process X is a geometric random walk if and only if $\log X$ is a random walk or, equivalently,

$$X(t) = \exp(Y(t))$$

for some random walk Y. Indeed, the random variables in (1.6) are identically distributed and independent of $\mathscr{F}_{t-1}$ if and only if this holds for

$$\Delta(\log X(t)) = \log X(t) - \log X(t-1) = \log\left(\frac{\Delta X(t)}{X(t-1)} + 1\right), \quad t \geq 1.$$

Random walks and geometric random walks represent processes of constant growth in an additive or multiplicative sense, respectively. Simple asset price models are often of geometric random walk type.

A **stopping time** τ is a random variable whose values are times, i.e. are in $\mathbb{N} \cup \{\infty\}$ in the discrete case. Additionally one requires that τ is consistent with the information structure $\mathbf{F}$. More precisely, one assumes that $\{\tau = t\} \in \mathscr{F}_t$ (or equivalently $\{\tau \leq t\} \in \mathscr{F}_t$) for any t. Intuitively, this means that the decision to say "stop!" right now can only be based on our current information. As an example consider the first time τ when an observed stock price hits the level 100. Even though this time is random and not known in advance, we obviously know τ in the instant it occurs. The situation is different if we define τ to be the instant *one period before the stock hits 100*. Since we cannot look into the future, we only know τ one period after it has happened. Consequently, this random variable is not a stopping time. Stopping times occur naturally in finance, e.g. in the context of American options, but they also play an important technical role in stochastic calculus.

As indicated above, the time when some adapted process first hits a given set is a stopping time:

Proposition 1.2 *Let X be some adapted process and B a Borel set. Then*

$$\tau := \inf\{t \geq 0 : X(t) \in B\}$$

is a stopping time.

Proof By adaptedness, we have $\{X(s) \in B\} \in \mathscr{F}_s \subset \mathscr{F}_t$, $s \leq t$ and hence

$$\{\tau \leq t\} = \bigcup_{s=0}^{t} \{X(s) \in B\} \in \mathscr{F}_t. \qquad \square$$

Occasionally, it turns out to be important to "freeze" a process at a stopping time. For any adapted process X at any stopping time τ, the **process stopped at** τ is defined as

$$X^\tau(t) := X(\tau \wedge t),$$

where we use the notation $a \wedge b := \min(a, b)$ as usual. The stopped process X^τ remains constant on the level $X(\tau)$ after time τ. It is easy to see that it is adapted as well.

The σ-field $\mathscr{F}_t$ represents the information up to time t. Sometimes we also need the concept of information up to τ, where τ now denotes a stopping time rather than a fixed number. The corresponding σ-field $\mathscr{F}_\tau$ can be interpreted as for fixed t: an event F belongs to $\mathscr{F}_\tau$ if we must wait at most until τ in order to decide whether F occurs or not. Formally, this σ-field is defined as

$$\mathscr{F}_\tau := \big\{F \in \mathscr{F} : F \cap \{\tau \leq t\} \in \mathscr{F}_t \text{ for any } t \geq 0\big\}.$$

Although it may not seem evident at first glance that this definition truly implements the above intuition, one can at least check that some intuitive properties hold:

Proposition 1.3 *Let σ, τ, τ_n be stopping times.*

1. *$\mathscr{F}_\tau$ is a σ-field.*
2. *If $\tau = t$ is a constant stopping time, $\mathscr{F}_\tau = \mathscr{F}_t$.*
3. *$\sigma \leq \tau$ implies that $\mathscr{F}_\sigma \subset \mathscr{F}_\tau$.*
4. *τ is $\mathscr{F}_\tau$-measurable.*
5. *Infima and suprema of finitely or countably many stopping times are again stopping times.*
6. *$\tau_n \downarrow \tau$ implies $\mathscr{F}_\tau = \bigcap_{n\in\mathbb{N}} \mathscr{F}_{\tau_n}$.*

Proof

1. It is straightforward to verify the axioms.
2. Since $\{\tau \leq s\} = \Omega$ for $s \leq t$ and $\varnothing$ for $s > t$, this follows immediately from the definition of $\mathscr{F}_\tau$.
3. For $F \in \mathscr{F}_\sigma$ we have $F \cap \{\tau \leq t\} = (F \cap \{\sigma \leq t\}) \cap \{\tau \leq t\} \in \mathscr{F}_t$.
4. We need to show that $\{\tau \leq s\} \in \mathscr{F}_\tau$ for any $s \geq 0$. Since $\{\tau \leq s\} \cap \{\tau \leq t\} = \{\tau \leq s \wedge t\} \in \mathscr{F}_{s\wedge t} \subset \mathscr{F}_t$ for any $t \geq 0$, this follows from the definition of $\mathscr{F}_\tau$.
5. For stopping times τ_i, $i \in I$ and $t \geq 0$ we have $\{\sup_{i\in I} \tau_i \leq t\} = \bigcap_{i\in I}\{\tau_i \leq t\} \in \mathscr{F}_t$ and $\{\inf_{i\in I} \tau_i \leq t\} = \bigcup_{i\in I}\{\tau_i \leq t\} \in \mathscr{F}_t$.

6. The inclusion $\subset$ follows from 3. If $F \in \mathscr{F}_{\tau_n}$ for all n, we have

$$F \cap \{\tau \leq t\} = \bigcup_{n \in \mathbb{N}} (F \cap \{\tau_n \leq t\}) \in \mathscr{F}_t,$$

which yields the claim. □

If X is a process and τ a stopping time, we denote by $X(\tau)$ the random variable $X(\omega, \tau(\omega))$. Since this does not make sense if $\tau(\omega) = \infty$, we consider $X(\tau)1_{\{\tau<\infty\}}$ if this may happen.

Proposition 1.4 *If X is an adapted process and τ a stopping time, $X(\tau)1_{\{\tau<\infty\}}$ is $\mathscr{F}_\tau$-measurable.*

Proof For Borel sets $B \subset \mathbb{R} \setminus \{0\}$ we have

$$\{X(\tau)1_{\{\tau<\infty\}} \in B\} \cap \{\tau \leq t\} = \bigcup_{s \leq t} \big(\{X(s) \in B\} \cap \{\tau = s\}\big) \in \mathscr{F}_t$$

as desired. □

The concept of martingales is central to stochastic calculus and finance. A **martingale** (resp. **submartingale**, **supermartingale**) is an adapted process X that is **integrable** in the sense that $E(|X(t)|) < \infty$ for any t and satisfies

$$E(X(t)|\mathscr{F}_s) = X(s) \quad (\text{resp. } \geq X(s), \leq X(s)) \tag{1.7}$$

for $s \leq t$. If X is a martingale, the best prediction for future values is the present level. For example, if the price process of an asset follows a martingale, it is neither going up nor down on average. In that sense it corresponds to a *fair game*. By contrast, submartingales (resp. supermartingales) may increase (resp. decrease) on average. They correspond to favourable (resp. unfavourable) games.

If ξ denotes an integrable random variable, then it naturally induces a martingale X, namely

$$X(t) = E(\xi|\mathscr{F}_t).$$

X is called the **martingale generated by** ξ. If the time horizon is finite, i.e. we consider the time set $\{0, 1, \ldots, T-1, T\}$ rather than $\mathbb{N}$, any martingale is generated by some random variable, namely by $X(T)$. This ceases to be true for infinite time horizons. For instance, random walks are not generated by a single random variable unless they are constant.

Example 1.5 (Density Process) A probability measure Q on $(\Omega, \mathscr{F})$ is called *equivalent* to P (written $Q \sim P$) if the events of probability 0 are the same under P and Q. By the Radon–Nikodym theorem, Q has a P-*density* and vice versa, i.e.

there are some unique random variables $\frac{dQ}{dP}, \frac{dP}{dQ}$ such that

$$Q(F) = E_P\left(1_F \frac{dQ}{dP}\right), \quad P(F) = E_Q\left(1_F \frac{dP}{dQ}\right)$$

for any set $F \in \mathscr{F}$, where E_P, E_Q denote expectation under P and Q, respectively. P, Q are in fact equivalent if and only if such mutual densities exist, in which case we have $\frac{dP}{dQ} = 1/\frac{dQ}{dP}$.

The martingale Z generated by $\frac{dQ}{dP}$ is called the **density process** of Q, i.e. we have

$$Z(t) = E_P\left(\frac{dQ}{dP}\middle| \mathscr{F}_t\right).$$

One easily verifies that $Z(t)$ coincides with the density of the restricted measures $Q|_{\mathscr{F}_t}$ relative to $P|_{\mathscr{F}_t}$, i.e. $Z(t)$ is $\mathscr{F}_t$-measurable and

$$Q(F) = E_P(1_F Z(t))$$

holds for any event $F \in \mathscr{F}_t$. Note further that Z and the density process Y of P relative to Q are reciprocal to each other because

$$Z(t) = \frac{dQ|_{\mathscr{F}_t}}{dP|_{\mathscr{F}_t}} = 1\Big/\frac{dP|_{\mathscr{F}_t}}{dQ|_{\mathscr{F}_t}} = 1/Y(t).$$

The density process Z can be used to compute conditional expectations relative to Q. Indeed, the **generalised Bayes' rule**

$$E_Q(\xi|\mathscr{F}_t) = \frac{E_P(\xi\frac{dQ}{dP}|\mathscr{F}_t)}{Z(t)}$$

(e.g. [226, Lemma 8.6.2], [154, III.3.9]) holds for sufficiently integrable random variables ξ because

$$\begin{aligned}
E_Q(\xi\zeta) &= E_P\left(\xi\zeta\frac{dQ}{dP}\right)\\
&= E_P\left(E_P\left(\xi\zeta\frac{dQ}{dP}\middle|\mathscr{F}_t\right)\right)\\
&= E_P\left(E_P\left(\frac{\xi\zeta\frac{dQ}{dP}}{Z(t)}\middle|\mathscr{F}_t\right)Z(t)\right)\\
&= E_Q\left(E_P\left(\frac{\xi\zeta\frac{dQ}{dP}}{Z(t)}\middle|\mathscr{F}_t\right)\right)\\
&= E_Q\left(\frac{E_P(\xi\frac{dQ}{dP}|\mathscr{F}_t)}{Z(t)}\zeta\right)
\end{aligned}$$

for any bounded $\mathscr{F}_t$-measurable ζ. Similarly, one shows

$$E_Q(\xi|\mathscr{F}_s) = \frac{E_P(\xi Z(t)|\mathscr{F}_s)}{Z(s)} \tag{1.8}$$

for $s \leq t$ and $\mathscr{F}_t$-measurable random variables ξ.

For later use, we note that a supermartingale with constant expectation is actually a martingale.

Proposition 1.6 *If X is a supermatingale and $T \geq 0$ with $E(X(T)) = E(X(0))$, then (1.7) holds with equality for any $s \leq t \leq T$.*

Proof The supermartingale property means that $E((X(t) - X(s))1_F) \leq 0$ for any $s \leq t$ and any $F \in \mathscr{F}_s$. Since

$$\begin{aligned}
0 &= E(X(T)) - E(X(0)) \\
&= E(X(T) - X(t)) + E\big((X(t) - X(s))1_F\big) + E\big((X(t) - X(s))1_{F^C}\big) \\
&\quad + E(X(s) - X(0)) \\
&\leq 0
\end{aligned}$$

for any $s \leq t \leq T$ and any event $F \in \mathscr{F}_s$, the four nonpositive summands must actually be 0. This yields $E((X(t) - X(s))1_F) = 0$ and hence the assertion. □

The following technical result is used in Sect. 1.5.

Lemma 1.7 *Let X be a supermartingale, Y a martingale, $t \leq T$, and $F \in \mathscr{F}_t$ with $X(t) = Y(t)$ on F and $X(T) \geq Y(T)$. Then $X(s) = Y(s)$ on F for $t \leq s \leq T$. The statement remains to hold if we only require $X - X^t$, $Y - Y^t$ instead of X, Y to be a supermartingale resp. martingale.*

Proof From

$$X(s) - Y(s) \geq E(X(T) - Y(T)|\mathscr{F}_s) \geq 0$$

and

$$E((X(s) - Y(s))1_F) \leq E((X(t) - Y(t))1_F) = 0$$

it follows that $(X(s) - Y(s))1_F = 0$. □

The concept of a martingale is "global" in the sense that (1.7) must be satisfied for any $s \leq t$. If we restrict attention to the case $s = t - 1$, we obtain the slightly more general "local" counterpart. A **local martingale** (resp. **submartingale**,

supermartingale) is an adapted process X which satisfies $E(|X(0)|) < \infty$, $E(|X(t)||\mathscr{F}_{t-1}) < \infty$ and

$$E(X(t)|\mathscr{F}_{t-1}) = X(t-1) \quad (\text{resp. } \geq X(t-1), \leq X(t-1)) \tag{1.9}$$

for $t = 1, 2, \ldots$ In discrete time the difference between martingales and local martingales is minor:

Proposition 1.8 *Any* integrable *local martingale (in the sense that $E(|X(t)|) < \infty$ for any t) is a martingale. An analogous statement holds for sub- and supermartingales.*

Proof This follows by induction from the law of iterated expectations. □

Integrability in Proposition 1.8 holds, for instance, if X is a nonnegative local supermartingale.

The above classes of processes are *stable under stopping* in the sense of the following proposition, which has a natural economic interpretation: you cannot turn a fair game into, say, a favourable one by stopping play at some reasonable time.

Proposition 1.9 *Let τ denote a stopping time. If X is a martingale (resp. sub-/supermartingale), so is X^τ. A corresponding statement holds for local martingales and local sub-/supermartingales.*

Proof We start by verifying the integrability conditions. For martingales (resp. sub-supermartingales) $E(|X(t)|) < \infty$ implies $E(|X^\tau(t)|) \leq E(\sum_{s=0}^t |X(s)|) < \infty$. For local martingales (resp. local sub-/supermartingales) $E(|X(t)||\mathscr{F}_{t-1}) < \infty$ yields

$$\begin{aligned} E(|X^\tau(t)||\mathscr{F}_{t-1}) &\leq \sum_{s=0}^{t} E(|X(s)||\mathscr{F}_{t-1}) \\ &= \sum_{s=0}^{t-1} |X(s)| + E(|X(t)||\mathscr{F}_{t-1}) < \infty. \end{aligned}$$

In order to verify (1.9), observe that $\{\tau \geq t\} = \{\tau \leq t-1\}^C \in \mathscr{F}_{t-1}$ implies

$$\begin{aligned} E(X^\tau(t)1_{\{\tau \geq t\}}|\mathscr{F}_{t-1}) &= E(X(t)1_{\{\tau \geq t\}}|\mathscr{F}_{t-1}) \\ &= E(X(t)|\mathscr{F}_{t-1})1_{\{\tau \geq t\}} \\ &= X(t-1)1_{\{\tau \geq t\}} \\ &= X^\tau(t-1)1_{\{\tau \geq t\}} \end{aligned}$$

(resp. $\geq$, $\leq$ in the sub-/supermartingale case). For $s < t$ we have $\{\tau = s\} \in \mathscr{F}_s \subset \mathscr{F}_{t-1}$ and hence

$$\begin{aligned} E(X^\tau(t)1_{\{\tau=s\}}|\mathscr{F}_{t-1}) &= E(X(s)1_{\{\tau=s\}}|\mathscr{F}_{t-1}) \\ &= X(s)1_{\{\tau=s\}} \\ &= X^\tau(t-1)1_{\{\tau=s\}}. \end{aligned}$$

Together we obtain

$$\begin{aligned} E(X^\tau(t)|\mathscr{F}_{t-1}) &= \sum_{s=0}^{t-1} E(X^\tau(t)1_{\{\tau=s\}}|\mathscr{F}_{t-1}) + E(X(t)1_{\{\tau\geq t\}}|\mathscr{F}_{t-1}) \\ &= \sum_{s=0}^{t-1} X^\tau(t-1)1_{\{\tau=s\}} + X^\tau(t-1)1_{\{\tau\geq t\}} = X^\tau(t-1) \end{aligned}$$

(resp. $\geq$, $\leq$). □

An alternative more complicated definition of local martingales uses stopping times, which turns out to be useful in continuous time.

Proposition 1.10 *An adapted process X is a local martingale if and only if there exists a sequence $(\tau_n)_{n\in\mathbb{N}}$ of stopping times, increasing to ∞ almost surely, such that the stopped processes X^{τ_n} are martingales for any n. A corresponding statement holds for sub-/supermartingales.*

Proof In order to show the *if* statement suppose that X^{τ_n} is a martingale for any n. Note that $\{\tau_n \geq t\} = \{\tau_n \leq t-1\}^C \in \mathscr{F}_{t-1}$ and integrability of $X^{\tau_n}(t)$ imply

$$\begin{aligned} E(|X(t)||\mathscr{F}_{t-1})1_{\{\tau_n\geq t\}} &= E(|X(t)|1_{\{\tau_n\geq t\}}|\mathscr{F}_{t-1}) \\ &= E(|X^{\tau_n}(t)|1_{\{\tau_n\geq t\}}|\mathscr{F}_{t-1}) < \infty \end{aligned}$$

and

$$\begin{aligned} E(X(t)|\mathscr{F}_{t-1})1_{\{\tau_n\geq t\}} &= E(X(t)1_{\{\tau_n\geq t\}}|\mathscr{F}_{t-1}) \\ &= E(X^{\tau_n}(t)1_{\{\tau_n\geq t\}}|\mathscr{F}_{t-1}) \\ &= E(X^{\tau_n}(t)|\mathscr{F}_{t-1})1_{\{\tau_n\geq t\}} \\ &= X^{\tau_n}(t-1)1_{\{\tau_n\geq t\}} \\ &= X(t-1)1_{\{\tau_n\geq t\}}. \end{aligned}$$

Therefore $E(|X(t)||\mathscr{F}_{t-1}) < \infty$ and (1.9) hold on $\cup_{n\geq 1}\{\tau_n \geq t\}$ and hence almost surely.

For the *only if* statement we define a sequence of stopping times $\tau_n := \inf\{t \geq 0 : E(|X(t+1)||\mathscr{F}_t) \geq n\}$. By Proposition 1.9, X^{τ_n} is a local martingale for any n. Since

$$\begin{aligned}
E(|X^{\tau_n}(t+1)|) &= E\big(|X(0)|1_{\{\tau_n=0\}}\big) + \sum_{s=0}^{t-1} E\big(|X(s+1)|1_{\{\tau_n=s+1\}}\big) \\
&\quad + E\big(|X(t+1)|1_{\{\tau_n>t\}}\big) \\
&\leq E(|X(0)|) + \sum_{s=0}^{t} E\big(|X(s+1)|1_{\{\tau_n\leq s\}^C}\big) \\
&= E(|X(0)|) + \sum_{s=0}^{t} E\big(E(|X(s+1)||\mathscr{F}_s)1_{\{\tau_n\leq s\}^C}\big) \\
&\leq E(|X(0)|) + (t+1)n < \infty,
\end{aligned}$$

X^{τ_n} is a martingale by Proposition 1.8. The assertion for sub-/supermartingales follows along the same lines. □

If X is a martingale, the martingale property (1.7) holds also for bounded and sometimes even general stopping times.

Theorem 1.11 (Doob's Stopping Theorem) *If X is a martingale, we have*

$$E(X(\tau)|\mathscr{F}_\sigma) = X(\sigma)$$

for any two bounded stopping times $\sigma \leq \tau$. If the martingale is generated by some random variable $X(\infty)$, we need not require σ, τ to be bounded.

For a supermartingale X, we have accordingly

$$E(X(\tau)|\mathscr{F}_\sigma) \leq X(\sigma)$$

for any two bounded stopping times $\sigma \leq \tau$.

Proof

Step 1: We show $E(X(r)|\mathscr{F}_\sigma) = X(\sigma)$ (resp. "$\leq$" in the supermartingale case). Indeed, $E(X(r)|\mathscr{F}_\sigma) = X(\sigma)$ means $E((X(r)-X(\sigma))1_F) = 0$ for any $F \in \mathscr{F}_\sigma$. For such F we have $F \cap \{\sigma = t\} = (F \cap \{\sigma \leq t\}) \setminus (F \cap \{\sigma \leq t-1\}) \in \mathscr{F}_t$ and hence $E((X(r) - X(\sigma))1_F) = \sum_{t=0}^{r} E((X(r) - X(t))1_{F\cap\{\sigma=t\}}) = 0$ because $X(t) = E(X(r)|\mathscr{F}_t)$.

Step 2: In view of Proposition 1.10, we can apply step 1 to the stopped process X^τ and obtain $E(X(\tau)|\mathscr{F}_\sigma) = E(X^\tau(r)|\mathscr{F}_\sigma) = X^\tau(\sigma) = X(\sigma)$ (resp. "$\leq$" in the supermartingale case). The statement for unbounded σ, τ follows for $r = \infty$. □

Sometimes one needs statements on uniform integrability of martingales. Here, inequalities such as the following one prove to be useful.

Theorem 1.12 (Doob's Inequality) *We have*

$$E\Big(\sup_{s\leq t} X(s)^2\Big) \leq 4E\big(X(t)^2\big), \quad t = 0, 1, 2, \dots$$

for any martingale X.

Proof The proof can be found, for example, in [275, Corollary VII.3.2]. □

One easily verifies that an integrable random walk X is a martingale if and only if the increments $\Delta X(t)$ have expectation 0. An analogous result holds for integrable geometric random walks whose relative increments $\Delta X(t)/X(t-1)$ have vanishing mean. For the martingale property to hold, one does not actually need the increments resp. relative increments of X to be identically distributed.

Martingales are expected to stay on the current level on average. More general processes may show an increasing, decreasing or possibly variable trend. This fact is expressed formally by a variant of *Doob's decomposition*. The idea is to decompose the increment $\Delta X(t)$ of an arbitrary process into a predictable trend component $\Delta A^X(t)$ and a random deviation $\Delta M^X(t)$ from this short time prediction.

Theorem 1.13 (Canonical Decomposition) *Any integrable adapted process X (i.e. with $E(|X(t)|) < \infty$ for any t) can be uniquely decomposed as*

$$X = X(0) + M^X + A^X \tag{1.10}$$

with some martingale M^X and some predictable process A^X satisfying $M^X(0) = A^X(0) = 0$. We call A^X the ***compensator*** *of X.*

Proof Define $A^X(t) = \sum_{s=1}^t \Delta A^X(s)$ by $\Delta A^X(s) := E(\Delta X(s)|\mathscr{F}_{s-1})$ and set $M^X := X - X(0) - A^X$. Predictability of A^X is obvious. The integrability of X implies that of A^X and thus of M^X. The latter is a martingale because

$$\begin{aligned} E(M^X(t)|\mathscr{F}_{t-1}) &= M^X(t-1) + E(\Delta X(t) - \Delta A^X(t)|\mathscr{F}_{t-1}) \\ &= M^X(t-1) + E(\Delta X(t)|\mathscr{F}_{t-1}) - E(E(\Delta X(t)|\mathscr{F}_{t-1})|\mathscr{F}_{t-1}) \\ &= M^X(t-1). \end{aligned}$$

Conversely, for any decomposition as in Theorem 1.13 we have

$$E(\Delta X(t)|\mathscr{F}_{t-1}) = E(\Delta M^X(t)|\mathscr{F}_{t-1}) + E(\Delta A^X(t)|\mathscr{F}_{t-1}) = \Delta A^X(t),$$

which means that it coincides with the decomposition in the first part of the proof. □

For instance, the compensator of an integrable random walk X equals

$$A^X(t) = \sum_{s=1}^{t} E(\Delta X(s)|\mathscr{F}_{s-1}) = tE(\Delta X(1)).$$

If X is a submartingale (resp. supermartingale), then A^X is increasing (resp. decreasing). This is the case commonly referred to as **Doob's decomposition**.

Note that uniqueness of the decomposition still holds if we require M^X to be only a *local* martingale. In this relaxed sense, it suffices to assume $E(|X(t)||\mathscr{F}_{t-1}) < \infty$ for any t in order to obtain (1.10) and to define the **compensator** A^X.

1.2 Stochastic Integration

Gains from trade in dynamic portfolios can be expressed in terms of *stochastic integrals*, which are nothing else than sums in discrete time. Consider an adapted process X and a predictable—or at least adapted—process φ. The **stochastic integral** of φ relative to X is the adapted process $\varphi \bullet X$ defined as

$$\varphi \bullet X(t) := \sum_{s=1}^{t} \varphi(s)\Delta X(s). \tag{1.11}$$

If both $\varphi = (\varphi_1, \dots, \varphi_d)$ and $X = (X_1, \dots, X_d)$ are vector-valued processes, we define $\varphi \bullet X$ to be the real-valued process given by

$$\varphi \bullet X(t) := \sum_{s=1}^{t} \sum_{i=1}^{d} \varphi_i(s)\Delta X_i(s). \tag{1.12}$$

In order to motivate this definition, let us interpret $X(t)$ as the price of a stock at time t. We invest in this stock using the trading strategy φ, i.e. $\varphi(t)$ denotes the number of shares we own at time t. Due to the price move from $X(t-1)$ to $X(t)$ our wealth changes by $\varphi(t)(X(t) - X(t-1)) = \varphi(t)\Delta X(t)$ in the period between $t-1$ and t. Consequently, the integral $\varphi \bullet X(t)$ stands for the cumulative gains from trade up to time t. If we invest in a portfolio of several stocks, both the trading strategy φ and the price process X are vector-valued. $\varphi_i(t)$ now stands for the number of shares of stock i and $X_i(t)$ is its price. In order to compute the total gains of the portfolio, we must sum up the gains $\varphi_i(t)\Delta X_i(t)$ in each single stock, which leads to (1.12).

For the above reasoning to make sense, one must be careful about the order in which things happen at time t. If $\varphi(t)(X(t) - X(t-1))$ is meant to stand for the gains at time t, we obviously have to buy the portfolio $\varphi(t)$ *before* prices change from $X(t-1)$ to $X(t)$. Put differently, we must choose $\varphi(t)$ at the end of period $t-1$, right after the stock price has attained the value $X(t-1)$. This choice can only be based on information up to time $t-1$ and in particular not on $X(t)$, which is as yet unknown. This motivates why one typically requires trading strategies to be predictable rather than adapted. The purely mathematical definition of $\varphi \bullet X$, however, makes sense regardless of any measurability assumption.

The **covariation process** $[X, Y]$ of adapted processes X, Y is defined as

$$[X, Y](t) := \sum_{s=1}^{t} \Delta X(s) \Delta Y(s). \tag{1.13}$$

Its compensator

$$\langle X, Y\rangle(t) = \sum_{s=1}^{t} E(\Delta X(s) \Delta Y(s) \,|\mathscr{F}_{s-1})$$

is called the **predictable covariation process** if it exists. In the special case $X = Y$ one refers to the **quadratic variation** resp. **predictable quadratic variation** of X. If X, Y are martingales, their predictable covariation can be viewed as a dynamic analogue of the covariance of two random variables.

We are now ready to state a few properties of stochastic integration:

Proposition 1.14 *For adapted processes X, Y, Z and predictable processes φ, ψ we have:*

1. *$\varphi \bullet X$ is linear in φ and X.*
2. *$[X, Y]$ and $\langle X, Y\rangle$ are symmetric and linear in X and Y.*
3. *$\psi \bullet (\varphi \bullet X) = (\psi\varphi) \bullet X$.*
4. *$[\varphi \bullet X, Y] = \varphi \bullet [X, Y]$.*
5. *$\langle \varphi \bullet X, Y\rangle = \varphi \bullet \langle X, Y\rangle$ whenever the predictable covariations are defined.*
6. ***(Integration by parts)***

$$\begin{aligned} XY &= X(0)Y(0) + X_- \bullet Y + Y \bullet X \\ &= X(0)Y(0) + X_- \bullet Y + Y_- \bullet X + [X, Y]. \end{aligned} \tag{1.14}$$

7. *If X is a local martingale, then so is $\varphi \bullet X$.*
8. *If $\varphi \geq 0$ and X is a local sub-/supermartingale, $\varphi \bullet X$ is a local sub-/supermartingale as well.*
9. *$A^{\varphi \bullet X} = \varphi \bullet A^X$ if the compensator A^X exists in the relaxed sense following Theorem 1.13.*
10. *If X, Y are martingales with $E(|X(t)Y(t)|) < \infty$ for any t, the process $XY - \langle X, Y\rangle$ is a martingale, i.e. $\langle X, Y\rangle$ is the compensator of XY.*
11. *$[X, [Y, Z]](t) = [[X, Y], Z](t) = \sum_{s=1}^{t} \Delta X(s) \Delta Y(s) \Delta Z(s)$.*

Proof

1. This is obvious from the definition.
2. This is obvious from the definition as well.
3. This follows from

$$\Delta(\psi \bullet (\varphi \bullet X))(t) = \psi(t)\Delta(\varphi \bullet X)(t) = \psi(t)\varphi(t)\Delta X(t) = \Delta((\psi\varphi) \bullet X)(t).$$

4. This follows from

$$\Delta[\varphi \bullet X, Y](t) = \varphi(t)\Delta X(t)\Delta Y(t) = \Delta(\varphi \bullet [X, Y])(t).$$

5. Predictability of φ yields

$$\begin{aligned}\Delta\langle\varphi \bullet X, Y\rangle(t) &= E(\Delta(\varphi \bullet X)(t)\Delta Y(t)\,|\mathscr{F}_{t-1}) \\ &= E(\varphi(t)\Delta X(t)\Delta Y(t)\,|\mathscr{F}_{t-1}) \\ &= \varphi(t)E(\Delta X(t)\Delta Y(t)\,|\mathscr{F}_{t-1}) \\ &= \Delta(\varphi \bullet \langle X, Y\rangle)(t).\end{aligned}$$

6. The first equation is

$$\begin{aligned}X(t)Y(t) &= X(0)Y(0) + \sum_{s=1}^{t}\big(X(s)Y(s) - X(s-1)Y(s-1)\big) \\ &= X(0)Y(0) + \sum_{s=1}^{t}\big(X(s-1)(Y(s) - Y(s-1))\big) \\ &\quad + \sum_{s=1}^{t}\big((X(s) - X(s-1))Y(s)\big) \\ &= X(0)Y(0) + \sum_{s=1}^{t}\big(X(s-1)\Delta Y(s) + Y(s)\Delta X(s)\big) \\ &= X(0)Y(0) + X_- \bullet Y(t) + Y \bullet X(t).\end{aligned}$$

The second follows from

$$Y \bullet X(t) = Y_- \bullet X(t) + (\Delta Y) \bullet X(t) = Y_- \bullet X(t) + [X, Y](t).$$

7. Predictability of φ and (1.9) yield

$$\begin{aligned} E(\varphi \bullet X(t)|\mathscr{F}_{t-1}) &= E(\varphi \bullet X(t-1) + \varphi(t)(X(t) - X(t-1))|\mathscr{F}_{t-1}) \\ &= \varphi \bullet X(t-1) + \varphi(t)(E(X(t)|\mathscr{F}_{t-1}) - X(t-1)) \\ &= \varphi \bullet X(t-1). \end{aligned}$$

8. This follows along the same lines as 7.
9. This follows from statement 7 because $\varphi \bullet X = \varphi \bullet M^X + \varphi \bullet A^X$ is the canonical decomposition of $\varphi \bullet X$.
10. This follows from statements 6 and 7.
11. This follows from the definition. □

If they make sense, the above rules hold for vector-valued processes as well, e.g.

$$\psi \bullet (\varphi \bullet X) = (\psi\varphi) \bullet X$$

if both φ, X are $\mathbb{R}^d$-valued.

Itō's formula is probably the most important rule in continuous-time stochastic calculus. This motivates why we state its simple discrete-time counterpart here.

Proposition 1.15 (Itō's Formula) *If X is an $\mathbb{R}^d$-valued adapted process and $f : \mathbb{R}^d \to \mathbb{R}$ a differentiable function, then*

$$\begin{aligned} f(X(t)) &= f(X(0)) + \sum_{s=1}^{t} (f(X(s)) - f(X(s-1))) \\ &= f(X(0)) + Df(X_-) \bullet X(t) \\ &\quad + \sum_{s=1}^{t} \Big(f(X(s)) - f(X(s-1)) - Df(X(s-1))^\top \Delta X(s) \Big), \end{aligned} \tag{1.15}$$

where $Df(x)$ denotes the derivative or gradient of f in x.

Proof The first statement is obvious. The second follows from the definition of the stochastic integral. □

If the increments $\Delta X(s)$ are small and f is sufficiently smooth, we may use the second-order Taylor expansion

$$\begin{aligned} f(X(s)) &= f(X(s-1) + \Delta X(s)) \\ &\approx f(X(s-1)) + f'(X(s-1))\Delta X(s) + \frac{1}{2} f''(X(s-1))(\Delta X(s))^2 \end{aligned}$$

in the univariate case, which leads to

$$\begin{aligned} f(X(t)) &\approx f(X(0)) + f'(X_-) \bullet X(t) + \sum_{s=1}^{t} \frac{1}{2} f''(X(s-1))(\Delta X(s))^2 \\ &= f(X(0)) + f'(X_-) \bullet X(t) + \frac{1}{2} f''(X_-) \bullet [X, X](t). \end{aligned}$$

If X is vector-valued, we obtain accordingly

$$f(X(t)) \approx f(X(0)) + Df(X_-) \bullet X(t) + \frac{1}{2} \sum_{i,j=1}^{d} D_{ij} f(X_-) \bullet [X_i, X_j](t). \tag{1.16}$$

Processes of multiplicative structure are called *stochastic exponentials*.

Definition 1.16 Let X be an adapted process. The unique adapted process Z satisfying

$$Z = 1 + Z_- \bullet X$$

is called the **stochastic exponential** of X and it is written $\mathscr{E}(X)$.

The stochastic exponential can be motivated from a financial point of view. Suppose that 1€ earns the possibly random interest ΔX_t in period t, i.e. 1€ at time $t-1$ turns into $(1 + \Delta X_t)$€ at time t. Then 1€ at time 0 runs up to $\mathscr{E}(X)(t)$ € at time t. It is easy to compute $\mathscr{E}(X)$ explicitly:

Proposition 1.17 *We have*

$$\mathscr{E}(X)(t) = \prod_{s=1}^{t} (1 + \Delta X(s)),$$

where the empty product for $t = 0$ is set to 1.

Proof For $Z(t) = \prod_{s=1}^{t}(1 + \Delta X(s))$ we have

$$\Delta Z(t) = Z(t) - Z(t-1) = Z(t-1)\Delta X(t)$$

and hence

$$Z(t) = Z(0) + \sum_{s=1}^{t} \Delta Z(s) = 1 + \sum_{s=1}^{t} Z(s-)\Delta X(s) = 1 + Z_- \bullet X(t). \qquad \square$$

We note in passing that $Z = c\mathscr{E}(X)$ is the unique solution to $Z = c + Z_- \bullet X$ for $c \in \mathbb{R}$.

The previous proposition implies that the stochastic exponential of a random walk $\widetilde{X}$ with increments $\Delta\widetilde{X}(t) > -1$ is a geometric random walk. More specifically, one can write any geometric random walk Z alternatively in exponential or stochastic exponential form, namely

$$Z = e^X = \mathscr{E}(\widetilde{X})$$

with random walks X, $\widetilde{X}$, respectively. X and $\widetilde{X}$ are related to each other via

$$\Delta\widetilde{X}(t) = e^{\Delta X(t)} - 1 \quad \text{resp.} \quad \Delta X(t) = \log(1 + \Delta\widetilde{X}(t)).$$

If the increments $\Delta X(s)$ are small enough, we can use the approximation

$$\log(1 + \Delta X(s)) \approx \Delta X(s) - \frac{1}{2}(\Delta X(s))^2$$

and obtain

$$\begin{aligned}
\mathscr{E}(X)(t) &= \prod_{s=1}^{t} (1 + \Delta X(s)) \\
&= \exp\left(\sum_{s=1}^{t} \log(1 + \Delta X(s))\right) \\
&\approx \exp\left(\sum_{s=1}^{t} \left(\Delta X(s) - \frac{1}{2}(\Delta X(s))^2\right)\right) \\
&= \exp\left(X(t) - X(0) - \frac{1}{2}[X, X](t)\right).
\end{aligned}$$

The product of stochastic exponentials is again a stochastic exponential. Observe the similarity of the following result to the rule $e^x e^y = e^{x+y}$ for the exponential function.

Proposition 1.18 (Yor's Formula)

$$\mathscr{E}(X)\mathscr{E}(Y) = \mathscr{E}(X + Y + [X, Y])$$

holds for any two adapted processes X, Y.

Proof Let $Z := \mathscr{E}(X)\mathscr{E}(Y)$. Integration by parts and the other statements of Proposition 1.14 yield

$$\begin{aligned} Z &= Z(0) + \mathscr{E}(X)_- \bullet \mathscr{E}(Y) + \mathscr{E}(Y)_- \bullet \mathscr{E}(X) + [\mathscr{E}(X), \mathscr{E}(Y)] \\ &= 1 + (\mathscr{E}(X)_-\mathscr{E}(Y)_-) \bullet Y + (\mathscr{E}(Y)_-\mathscr{E}(X)_-) \bullet X + (\mathscr{E}(X)_-\mathscr{E}(Y)_-) \bullet [X, Y] \\ &= 1 + Z_- \bullet (X + Y + [X, Y]), \end{aligned}$$

which implies that $Z = \mathscr{E}(X + Y + [X, Y])$. □

If an adapted process Z does not attain the value 0, it can be written as

$$Z = Z(0)\mathscr{E}(X)$$

with some unique process X satisfying $X(0) = 0$. This process X is naturally called the **stochastic logarithm** $\mathscr{L}(Z)$ of Z. We have

$$\mathscr{L}(Z) = \frac{1}{Z_-} \bullet Z.$$

Indeed, $X = \frac{1}{Z_-} \bullet Z$ satisfies

$$\frac{Z_-}{Z(0)} \bullet X = \frac{Z_-}{Z(0)} \bullet \left(\frac{1}{Z_-} \bullet Z\right) = \frac{1}{Z(0)} \bullet Z = \frac{Z}{Z(0)} - 1$$

and hence

$$\frac{Z}{Z(0)} = \mathscr{E}(X)$$

as claimed. Observe that the same notation $\mathscr{L}(Z)$ is used for the stochastic logarithm of a process and for the law of a random variable Z. It should be evident from the context which one we are referring to.

Changes of the underlying probability measure play an important role in Mathematical Finance. Since the notion of a martingale involves expectation, it is not invariant under such measure changes.

Proposition 1.19 *Let $Q \sim P$ be a probability measure with density process Z. An adapted process X is a Q-martingale (resp. Q-local martingale) if and only if XZ is a P-martingale (resp. P-local martingale).*

Proof X is a Q-local martingale if and only if

$$E_Q(X(t)|\mathscr{F}_{t-1}) = X(t-1) \tag{1.17}$$

for any t. By Bayes' rule (1.8) the left-hand side equals $E(X(t)Z(t)|\mathscr{F}_{t-1})/Z(t-1)$. Hence (1.17) is equivalent to $E(X(t)Z(t)|\mathscr{F}_{t-1}) = X(t-1)Z(t-1)$, which is the local martingale property of XZ relative to P. The integrability property for martingales (cf. Proposition 1.8) is shown similarly. □

A martingale X may possibly show a trend under the new probability measure Q. This trend can be expressed in terms of a predictable covariation.

Proposition 1.20 *Let $Q \sim P$ be a probability measure with density process Z. Moreover, suppose that X is a P-martingale. If X is Q-integrable, its Q-compensator equals $\langle \mathscr{L}(Z), X\rangle$, where the angle bracket is computed relative to P.*

Proof Denote the Q-compensator of X by A. In view of the comment after Theorem 1.13, we have that $X - X(0) - A$ is a Q-local martingale. By Proposition 1.19 this means that $Z(X - X(0) - A)$ is a P-local martingale. Integration by parts yields

$$Z(X-X(0)-A) = Z_- \bullet X + (X-X(0))_- \bullet Z + [Z, X-X(0)] - Z_- \bullet A - A \bullet Z.$$

The integrals relative to X and Z are local martingales. Hence

$$[Z, X] - Z_- \bullet A = [Z, X - X(0)] - Z_- \bullet A$$

is a P-local martingale as well, which implies $Z_- \bullet A = \langle Z, X\rangle$. Consequently, $A = \frac{1}{Z_-} \bullet \langle Z, X\rangle = \langle \mathscr{L}(Z), X\rangle$ as desired. □

The continuous-time analogue of the following representation theorem plays an important role in Mathematical Finance.

Proposition 1.21 (Martingale Representation) *Suppose that X is a random walk such that the increments $\Delta X(t)$ have only two values $a, -b$, attained with probabilities p and $1-p$, respectively. If X is a martingale and if the filtration is generated by X, any martingale M can be written as a stochastic integral $M = M(0) + \varphi \bullet X$ with some predictable process φ.*

Proof The martingale property of X implies $ap - b(1-p) = 0$. Since $\Delta M(t)$ is $\sigma(\Delta X(1), \dots, \Delta X(t))$-measurable, there is some function $f_t : \{-b, a\}^t \to \mathbb{R}$ such that $\Delta M(t) = f_t(\Delta X(1), \dots, \Delta X(t))$. The martingale property of M and (1.4) yield

$$\begin{aligned} 0 &= E(\Delta M(t)|\mathscr{F}_{t-1}) \\ &= p f_t(\Delta X(1), \dots, \Delta X(t-1), a) + (1-p) f_t(\Delta X(1), \dots, \Delta X(t-1), -b) \end{aligned}$$

and hence

$$\frac{1}{a} f_t(\Delta X(1), \dots, \Delta X(t-1), a) = -\frac{1}{b} f_t(\Delta X(1), \dots, \Delta X(t-1), -b) =: \varphi(t).$$

This implies

$$\varphi(t)\Delta X(t) = f_t(\Delta X(1), \dots, \Delta X(t-1), a) = \Delta M(t)$$

for $\Delta X(t) = a$ and likewise for $\Delta X(t) = -b$. □

The following statement and its continuous-time counterpart play a key role in Mathematical Finance.

Theorem 1.22 (Optional Decomposition) *Let S denote an $\mathbb{R}^d$-valued process and $\mathscr{Q}$ the set of all $Q \sim P$ such that S is a Q-local martingale (in the sense that $S_1, \dots, S_d$ are Q-local martingales). Moreover, let X be a process which is a Q-local supermartingale relative to all $Q \in \mathscr{Q}$. If $\mathscr{Q}$ is not empty, there exists a predictable $\mathbb{R}^d$-valued process φ such that $C := X(0) + \varphi \bullet S - X$ is increasing.*

Proof This is stated in [114, Theorem 2] if the time horizon is finite, i.e. if S and X are constant after some deterministic time $T < \infty$. The general case follows from [114, Theorem 1].

We illustrate the statement in the case of a finite time horizon $T \in \mathbb{N}$ and a finite sample space Ω whose elements have strictly positive probability.

Step 1: The proof will be based on an application of the separating hyperplane theorem A.14. To this end, let U be the finite-dimensional space of adapted processes $x = (x(t))_{t=0,\dots,T}$ with $x(0) = 0$. Moreover, we consider $V := U$ as the dual space of U via $y(x) := E(\sum_{t=1}^T x(t)y(t))$. Set

$$K := \left\{(\varphi(t)^\top \Delta S(t))_{t=0,\dots T} : \varphi \text{ predictable and } \mathbb{R}^d\text{-valued}\right\}$$

and

$$M := \Bigg\{\big(\lambda\Delta X(t)+(1-\lambda)x(t)\big)_{t=0,\dots,T} : \lambda \in [0,1], (x(t))_{t=0,\dots,T} \text{ nonnegative adapted process with } x(0) = 0 \text{ and } E\Bigg(\sum_{t=1}^T x(t)\Bigg) = 1\Bigg\}.$$

One easily verifies that K is a subspace and M a compact convex subset of U.

Step 2: In steps 2–4 we show by contradiction that $K \cap M \neq \varnothing$. Otherwise the separating hyperplane theorem A.14 yields the existence of $y \in V$ with

$$E\Bigg(\sum_{t=1}^T x(t)y(t)\Bigg) = 0, \quad x \in K, \tag{1.18}$$

$$E\Bigg(\sum_{t=1}^T x(t)y(t)\Bigg) > 0, \quad x \in M. \tag{1.19}$$

Since $x \in M$ for

$$x(s) := \begin{cases} 1_F/E(F) & \text{for } s = t, \\ 0 & \text{otherwise} \end{cases}$$

with fixed $t \in \{1, \dots, T\}$ and $F \in \mathscr{F}_t$, we conclude from (1.19) that y is strictly positive. Define the martingale

$$N(t) := \sum_{s=1}^{t} \left(\frac{y(s)}{E(y(s)|\mathscr{F}_{s-1})} - 1 \right)$$

and $Z = \mathscr{E}(N)$. Since $\Delta N > -1$, we have that Z is the density process of some probability measure $Q \sim P$.

Step 3: We show that $Q \in \mathscr{Q}$. By Proposition 1.19 this follows if $(\varphi \bullet S)Z$ is a martingale for any predictable $\mathbb{R}^d$-valued φ. Defining the predictable, positive process $a(t) := E(y(t)|\mathscr{F}_{t-1})$, $t = 1, \dots, T$, it suffices to show that $V := \frac{a}{Z_-} \bullet ((\varphi \bullet S)Z)$ is a martingale, cf. Proposition 1.14(7). Fix $t \in \{0, \dots, T\}$ and $F \in \mathscr{F}_t$. Since

$$(\varphi \bullet S)Z = (\varphi \bullet S)_- \bullet Z + Z_- \bullet (\varphi \bullet S) + [\varphi \bullet S, Z]$$

and $Z = \mathscr{E}(N)$, we have

$$(\varphi \bullet S)Z = Z_- \bullet \big((\varphi \bullet S)_- \bullet N + \varphi \bullet S + [\varphi \bullet S, N]\big). \tag{1.20}$$

Hence V equals

$$\begin{aligned} a \bullet \big(\varphi \bullet S + [\varphi \bullet S, N]\big)(t) &= \sum_{s=1}^{t} a(s)\bigg(\varphi(s)^\top \Delta S(s) + \varphi(s)^\top \Delta S(s)\bigg(\frac{y(s)}{a(s)} - 1\bigg)\bigg) \\ &= \sum_{s=1}^{t} \varphi(s)^\top \Delta S(s) y(s) =: \widetilde{V}(t) \end{aligned}$$

up to a martingale. Observe that

$$\begin{aligned} E\Big((\widetilde{V}(T) - \widetilde{V}(t))1_F\Big) &= E\bigg(\sum_{s=t+1}^{T} \varphi(s)^\top \Delta S(s) y(s) 1_F\bigg) \\ &= E\bigg(\sum_{s=1}^{T} \widetilde{\varphi}(s)^\top \Delta S(s) y(s)\bigg) \end{aligned} \tag{1.21}$$

for the predictable process

$$\widetilde{\varphi}(s) := \begin{cases} \varphi(s)1_F & \text{for } s > t, \\ 0 & \text{for } s \leq t. \end{cases}$$

By (1.18) we have that (1.21) equals 0, which in turn means that $\widetilde{V}$ and hence also V are martingales as desired.

Step 4: By assumption X is a Q-supermartingale. Since

$$X(s)Z(s) \geq E_Q(X(t)|\mathscr{F}_s)Z(s) = E(X(t)Z(t)|\mathscr{F}_s)$$

for $s \leq t$ by (1.8), we have that XZ is a supermartingale relative to P. By Proposition 1.14(8) this in turn implies that $\widetilde{S} := \frac{a}{Z_-} \bullet (XZ)$ is a supermartingale. Essentially the same calculations as in (1.20–1.21) yield

$$0 \geq E(\widetilde{S}(T) - \widetilde{S}(0)) = E\left(\sum_{s=1}^{T} \Delta X(s)y(s)\right)$$

in contradiction to (1.19). It follows that $K \cap M = \varnothing$ cannot hold, cf. step 2.

Step 5: Since $K \cap M \neq \varnothing$, there exists some predictable φ, some $\lambda \in [0, 1]$ and some nonnegative adapted x with $x(0) = 0$ and $P(\sum_{t=1}^{T} x(t) > 0) > 0$ such that

$$\lambda \Delta X(t) + (1 - \lambda)x(t) = \varphi(t)^\top \Delta S(t), \quad t = 1, \dots, T. \tag{1.22}$$

Assume by contradiction that $\lambda = 0$. This implies $\sum_{t=1}^{T} x(t) = \varphi \bullet S(T)$. For any $Q \in \mathscr{Q}$ we have $E_Q(\varphi \bullet S(T)) = 0$. On the other hand, $\sum_{t=1}^{T} x(t) \geq 0$ and $Q(\sum_{t=1}^{T} x(t) > 0) > 0$ imply that $E_Q(\varphi \bullet S(T)) > 0$ and hence a contradiction to (1.19). Consequently $\lambda > 0$.

Step 6: From (1.22) and $\lambda > 0$ we obtain

$$\Delta X(t) = \frac{\varphi(t)^\top}{\lambda} \Delta S(t) - \frac{1 - \lambda}{\lambda} x(t), \quad t = 1, \dots T,$$

whence $X = X(0) + \frac{\varphi}{\lambda} \bullet S - C$ for the increasing process $C(t) = \sum_{s=1}^{t} \frac{1-\lambda}{\lambda} x(s)$. This yields the assertion. □

A prime example for a process X as in Theorem 1.22 is given by

$$X(t) = \operatorname*{ess\,sup}_{Q \in \mathscr{Q}} E_Q(H|\mathscr{F}_t), \quad t \geq 0, \tag{1.23}$$

where H denotes a bounded random variable.

Proposition 1.23 *X in (1.23) is a Q-supermartingale for all $Q \in \mathscr{Q}$.*

Proof

Step 1: Fix $t \geq 0$. We show that $(E_Q(H|\mathscr{F}_t))_{Q\in\mathscr{Q}}$ has the lattice property in the sense of Lemma A.3. To this end, take $Q_1, Q_2 \in \mathscr{Q}$ with density processes Y_1, Y_2 and consider $F := \{E_{Q_1}(H|\mathscr{F}_t) \leq E_{Q_2}(H|\mathscr{F}_t)\} \in \mathscr{F}_t$. Using Proposition 1.19 it is easy to verify that

$$\widetilde{Y}(s) := \begin{cases} Y_1(s) & \text{for } s \leq t, \\ Y_1(s) & \text{on } F^C \text{ for } s > t, \\ \frac{Y_1(t)}{Y_2(t)} Y_2(s) & \text{on } F \text{ for } s > t \end{cases}$$

is the density process of some probability measure $\widetilde{Q} \in \mathscr{Q}$. Since

$$\begin{aligned} E_{\widetilde{Q}}(H|\mathscr{F}_t) &= E_{Q_1}(H|\mathscr{F}_t)1_{F^C} + E_{Q_2}(H|\mathscr{F}_t)1_F \\ &= E_{Q_1}(H|\mathscr{F}_t) \vee E_{Q_2}(H|\mathscr{F}_t), \end{aligned}$$

the lattice property holds.

Step 2: Fix $Q \in \mathscr{Q}$ and $t \geq 0$. By Lemma A.3 there is a sequence of probability measures $Q_n \in \mathscr{Q}$ such that $E_{Q_n}(H|\mathscr{F}_t) \uparrow \operatorname{ess\,sup}_{\widetilde{Q}\in\mathscr{Q}} E(H|\mathscr{F}_t)$ as $n \to \infty$. Denote by Y_n and Y the density processes of Q_n and Q, respectively. Using Proposition 1.19 and the generalised Bayes' rule (1.8) it is easy to verify that the process

$$\widetilde{Y}_n(s) := \begin{cases} Y(s) & \text{for } s \leq t, \\ \frac{Y(t)}{Y_n(t)} Y_n(s) & \text{for } s > t \end{cases}$$

is the density process of some probability measure $\widetilde{Q}_n \in \mathscr{Q}$ satisfying $\widetilde{Q}_n|_{\mathscr{F}_t} = Q|_{\mathscr{F}_t}$ and $E_{\widetilde{Q}_n}(H|\mathscr{F}_t) = E_{Q_n}(H|\mathscr{F}_t)$. Together with the monotone convergence theorem for conditional expectations we conclude that

$$\begin{aligned} E_Q(X(t)|\mathscr{F}_s) &= E_Q\left(\operatorname*{ess\,sup}_{\widetilde{Q}\in\mathscr{Q}} E_{\widetilde{Q}}(H|\mathscr{F}_t) \middle| \mathscr{F}_s \right) \\ &= E_Q\left(\sup_{n\in\mathbb{N}} E_{Q_n}(H|\mathscr{F}_t) \middle| \mathscr{F}_s \right) \\ &= \sup_{n\in\mathbb{N}} E_Q\big(E_{Q_n}(H|\mathscr{F}_t)\big|\mathscr{F}_s\big) \\ &= \sup_{n\in\mathbb{N}} E_{\widetilde{Q}_n}\left(E_{\widetilde{Q}_n}(H|\mathscr{F}_t) \middle| \mathscr{F}_s \right) \end{aligned}$$

$$
\begin{aligned}
&= \sup_{n\in\mathbb{N}} E_{\widetilde{Q}_n}(H|\mathscr{F}_s) \\
&\leq \operatorname*{ess\,sup}_{\widetilde{Q}\in\mathscr{Q}} E_{\widetilde{Q}}(H|\mathscr{F}_s) \\
&= X(s)
\end{aligned}
$$

for $s \leq t$. □

1.3 Jump Characteristics

The distribution of a concrete stochastic process *as a whole* is often unknown in the first place. However, we usually have a local conception of its dynamics, i.e. at time $t-1$ we are aware of the distribution of the next value $X(t)$ given the history up to $t-1$. This information can be used in a second step to derive unconditional expected values $E(X(t))$, probabilities etc. We consider two concepts of local descriptions of the process, jump characteristics in the present section and generators of Markov processes in the next.

If X is an adapted process with values in $E \subset \mathbb{R}^d$, we call the mapping

$$K^X(t, B) := P(\Delta X(t) \in B|\mathscr{F}_{t-1}) := E\big(1_B(\Delta X(t))\big|\mathscr{F}_{t-1}\big) \tag{1.24}$$

for $t = 1, 2, \dots$ and $B \in \mathscr{B}(E)$ the **jump characteristic** of X. It is nothing else than the conditional law of the increments of the process. The name is inspired by *semimartingale characteristics*. This more involved continuous-time counterpart is discussed in Chap. 4.

In applications the jump characteristic depends only on the present value of the process. Specifically, we say an adapted process X is **of Markov type** if $K^X(t, B)$ depends on ω, t only through $X(t-1)(\omega)$, i.e. more precisely if it is of the form

$$K^X(t, B) = \kappa(X(t-1), B) \tag{1.25}$$

with some function κ that is a probability measure in its second argument. This Markovian case is studied more thoroughly in the next section.

Random walks have particularly simple jump characteristics.

Proposition 1.24 (Random Walk) *An adapted process X with $X(0) = 0$ is a random walk if and only if its jump characteristic is of the form*

$$K^X(t, B) = \nu(B)$$

for some probability measure ν which does not depend on (ω, t). In this case ν is the law of $\Delta X(1)$. Note that X is of Markov type.

Proof If X is a random walk, we have $P^{\Delta X(t)|\mathscr{F}_{t-1}} = P^{\Delta X(t)} = P^{\Delta X(1)}$ since $\Delta X(t)$ is independent of $\mathscr{F}_{t-1}$ and has the same law for all t.

Conversely,

$$\begin{aligned} P(\{\Delta X(t) \in B\} \cap F) &= E(1_B(\Delta X(t))1_F) \\ &= \int E\big(1_B(\Delta X(t))\big|\mathscr{F}_{t-1}\big)1_F dP \\ &= \int \nu(B)1_F dP \\ &= \nu(B)P(F) \end{aligned}$$

for $F \in \mathscr{F}_{t-1}$. For $F = \Omega$ we obtain $P(\Delta X(t) \in B) = \nu(B) = P(\Delta X(1) \in B)$. Hence $\Delta X(t)$ is independent of $\mathscr{F}_{t-1}$ and has the same law for all t. □

Geometric random walks turn out to be of Markov type as well.

Example 1.25 The jump characteristic of a geometric random walk X is given by

$$K^X(t, B) = \varrho\big(\{x \in \mathbb{R}^d : X(t-1)(x-1) \in B\}\big),$$

where ϱ denotes the distribution of $X(1)/X(0) = X(1)$. Indeed, we have

$$\begin{aligned} E\big(1_B(\Delta X(t))\big|\mathscr{F}_{t-1}\big) &= E\left(1_B\left(X(t-1)\left(\frac{X(t)}{X(t-1)} - 1\right)\right)\middle|\mathscr{F}_{t-1}\right) \\ &= \int 1_B(X(t-1)(x-1))\varrho(dx) \end{aligned}$$

by (1.4) and the fact that $X(t)/X(t-1)$ has law ϱ and is independent of $\mathscr{F}_{t-1}$.

Since adapted processes are invariant under stochastic integration, stopping, application of continuous mappings and measure changes, it makes sense to discuss the effect of these operations on the jump characteristic. The following propositions are provided as a motivation for similar rules in Chap. 4.

Proposition 1.26 (Stochastic Integration) *If X is an adapted process with jump characteristic K^X and φ is a predictable process, the jump characteristic $K^{\varphi \bullet X}$ of $\varphi \bullet X$ is given by*

$$K^{\varphi \bullet X}(t, B) = \int 1_B(\varphi(t)x)K^X(t, dx).$$

Proof By (1.3) we have

$$E\left(1_B(\Delta(\varphi \bullet X(t)))\middle|\mathscr{F}_{t-1}\right) = E\left(1_B(\varphi(t)\Delta X(t))\middle|\mathscr{F}_{t-1}\right)$$
$$= \int 1_B(\varphi(t)x)P^{\Delta X(t)|\mathscr{F}_{t-1}}(dx),$$

which yields the claim because $P^{\Delta X(t)|\mathscr{F}_{t-1}} = K^X(t,\cdot)$. □

Proposition 1.27 (Stopping) *If X is an adapted process with jump characteristic K^X and τ is a stopping time, the jump characteristic K^{X^τ} of X^τ is given by*

$$K^{X^\tau}(t,B) = K^X(t,B)1_{\{\tau\geq t\}} + \varepsilon_0(B)1_{\{\tau<t\}}.$$

Here, ε_0 denotes the Dirac measure in 0.

Proof Since $X^\tau = X(0) + \varphi \bullet X$ with $\varphi(t) := 1_{\{\tau\geq t\}}$, the assertion follows from the previous proposition. □

Proposition 1.28 (Functions) *If X is an adapted process with jump characteristic K^X and f is a real- or vector-valued function, then the jump characteristic $K^{f(X)}$ of the process $f(X)$ is given by*

$$K^{f(X)}(t,B) = \int 1_B\big(f(X(t-1)+x) - f(X(t-1))\big)K^X(t,dx).$$

Proof By (1.3) we have

$$E\left(1_B(f(X(t)) - f(X(t-1)))\middle|\mathscr{F}_{t-1}\right)$$
$$= \int 1_B\big(f(X(t-1)+x) - f(X(t-1))\big)P^{\Delta X(t)|\mathscr{F}_{t-1}}(dx)$$
$$= \int 1_B\big(f(X(t-1)+x) - f(X(t-1))\big)K^X(t,dx),$$

which yields the claim. □

Proposition 1.29 (Change of Measure) *Suppose that $Q \sim P$ denotes a probability measure with density process $Z = Z(0)\mathscr{E}(N)$. Let X be an adapted process and denote by $K^{(X,N)}$ the characteristic of the bivariate process (X,N). Then the jump characteristic $K^{X,Q}$ of X relative to Q is given by*

$$K^{X,Q}(t,B) = \int 1_B(x)(1+y)K^{(X,N)}(t,d(x,y)).$$

Proof By (1.8) we have

$$\begin{aligned}E_Q\big(1_B(\Delta X(t))\big|\mathscr{F}_{t-1}\big) &= E\big(1_B(\Delta X(t))\tfrac{Z(t)}{Z(t-1)}\big|\mathscr{F}_{t-1}\big)\\ &= E\big(1_B(\Delta X(t))(1+\Delta N(t))\big|\mathscr{F}_{t-1}\big).\end{aligned}$$

Note that

$$E\big(f(\Delta X(t),\Delta N(t))\big|\mathscr{F}_{t-1}\big) = \int f(x,y)K^{(X,N)}(t,d(x,y))$$

holds by definition for indicator functions $f(x,y) = 1_C(x,y)$ and hence by standard arguments for arbitrary functions f, cf. the proof of Proposition 1.38. Considering $f(x,y) = 1_C(x)(1+y)$ yields the claim. □

Propositions 1.26–1.28 are easy to memorise: if x stands for the jump size of X at t, then $\varphi \bullet X$ jumps by $\varphi(t)x$ (needed for Proposition 1.26), X^τ jumps by x if $\tau \geq t$ and by 0 if $\tau < t$ (needed for Proposition 1.27), and $f(X)$ by $f(X(t-1)+x) - f(X(t-1))$ (for Proposition 1.28). Proposition 1.29 may seem less obvious. It can be understood by viewing the transition from P to Q as a composition of one-period measure changes with conditional density $1+\Delta N(t)$.

In order to illustrate some of these rules, we consider the following

Example 1.30 We have already observed that geometric random walks can be written as ordinary or alternatively as stochastic exponentials of random walks, i.e.

$$Z = \exp(X) = \mathscr{E}(\widetilde{X}).$$

This can also be seen by computing their characteristics. Proposition 1.28 for $f(x) = e^x$ yields

$$\begin{aligned}K^Z(t,B) &= \int 1_B\big(e^{X(t-1)+x} - e^{X(t-1)}\big)K^X(t,dx)\\ &= \int 1_B\big(Z(t-1)(e^x-1)\big)K^X(t,dx),\end{aligned}$$

where K^X, K^Z denote the jump characteristics of X and Z, respectively. Since $\widetilde{X} = \frac{1}{Z_-} \bullet Z$, we have

$$\begin{aligned}K^{\widetilde{X}}(t,B) &= \int 1_B(x/Z(t-1))K^Z(t,dx)\\ &= \int 1_B(e^x-1)K^X(t,dx) \qquad (1.26)\end{aligned}$$

for the jump characteristic of $\widetilde{X}$ by Proposition 1.26. From Proposition 1.24 we observe that $\widetilde{X}$ is a random walk if and only if this holds for X.

Conversely, we could also have applied Proposition 1.26 to $Z = Z(0) + Z_- \bullet \widetilde{X}$ to obtain

$$K^Z(t, B) = \int 1_B\big(Z(t-1)x\big)K^{\widetilde{X}}(t, dx).$$

Proposition 1.28 for $f(x) = \log x$ now yields

$$\begin{aligned} K^X(t, B) &= \int 1_B\big(\log(Z(t-1)+z) - \log Z(t-1)\big)K^Z(t, dz) \\ &= \int 1_B(\log(1+x))K^{\widetilde{X}}(t, dx), \end{aligned}$$

which is equivalent to (1.26).

For the following we define the **identity process** I as

$$I(t) := t.$$

The characteristics can be used to compute the compensator of an adapted process.

Proposition 1.31 (Compensator) *If X is an adapted process, its compensator A^X and its jump characteristic K^X are related to each other via*

$$A^X = a^X \bullet I$$

with

$$a^X(t) := \int x K^X(t, dx)$$

provided that X is integrable or, more generally, if the integral a^X is finite.

Proof By definition of the compensator we have

$$\begin{aligned} \Delta A^X(t) &= E(\Delta X(t)|\mathscr{F}_{t-1}) = \int x P^{\Delta X(t)|\mathscr{F}_{t-1}}(dx) \\ &= \int x K^X(t, dx) = a^X(t) = \Delta(a^X \bullet I)(t). \end{aligned}$$ □

Since the predictable covariation is a compensator, it can also be expressed in terms of characteristics.

Proposition 1.32 (Predictable Covariation) *The predictable covariations of two adapted processes X, Y and of their martingale parts M^X, M^Y are given by*

$$\langle X, Y\rangle = \widetilde{c}^{X,Y} \bullet I,$$
$$\langle M^X, M^Y\rangle = \widehat{c}^{X,Y} \bullet I$$

with

$$\widetilde{c}^{X,Y}(t) := \int xyK^{(X,Y)}(t, d(x, y)),$$
$$\widehat{c}^{X,Y}(t) := \int xyK^{(X,Y)}(t, d(x, y)) - a^X(t)a^Y(t),$$

provided that the integrals $a^X, a^Y, \widetilde{c}^{X,Y}$ are finite.

Proof The first statement follows similarly as Proposition 1.31 by observing that

$$\Delta\langle X, Y\rangle(t) = E(\Delta X(t)\Delta Y(t)|\mathscr{F}_{t-1}).$$

The second in turn follows from the first and from Proposition 1.31 because

$$\begin{aligned}\Delta\langle M^X, M^Y\rangle(t) &= E(\Delta M^X(t)\Delta M^Y(t)|\mathscr{F}_{t-1})\\ &= E(\Delta X(t)\Delta Y(t)|\mathscr{F}_{t-1}) - E(\Delta X(t)|\mathscr{F}_{t-1})\Delta A^Y(t)\\ &\quad - \Delta A^X(t)E(\Delta Y(t)|\mathscr{F}_{t-1}) + \Delta A^X(t)\Delta A^Y(t)\\ &= \Delta\langle X, Y\rangle(t) - \Delta A^X(t)\Delta A^Y(t)\\ &= \int xyK^{(X,Y)}(t, d(x, y)) - a^X(t)a^Y(t).\end{aligned}$$ □

Let us rephrase the integration by parts rule in terms of characteristics.

Proposition 1.33 *For adapted processes X, Y we have*

$$a^{XY}(t) = X(t-1)a^Y(t) + Y(t-1)a^X(t) + \widetilde{c}^{X,Y}(t),$$

provided that X, Y and XY are integrable or, more generally, if the integrals $a^X, a^Y, \widetilde{c}^{X,Y}$ are finite.

Proof Computing the compensators of

$$XY = X(0)Y(0) + X_- \bullet Y + Y_- \bullet X + [X, Y]$$

yields

$$\begin{aligned} a^{XY} \bullet I &= A^{XY} \\ &= X_- \bullet A^Y + Y_- \bullet A^X + A^{[X,Y]} \\ &= (X_- a^Y) \bullet I + (Y_- a^X) \bullet I + \tilde{c}^{X,Y} \bullet I \end{aligned}$$

by Propositions 1.14(3, 9) and 1.32. Considering increments yields the claim. □

1.4 Markov Processes

Fix a state space $E \subset \mathbb{R}^d$ containing the possible values of the stochastic processes under consideration. We call an adapted E-valued process X a **(homogeneous) Markov process** if

$$P(X(s+t) \in B | \mathscr{F}_s) = P(t, X(s), B), \quad s, t \geq 0, \; B \in \mathscr{B}(E) \tag{1.27}$$

holds for some family $(P(t, x, \cdot))_{t \geq 0, x \in E}$ of probability measures on E. Intuitively, this means that the future evolution $X(s+t)$ given the past up to s depends on this past only through the present value $X(s)$. In this sense the process has *no memory*. Moreover, its dynamics do not depend explicitly on time. If the process attains the value x at time t, it evolves in the future as if it had started afresh at this point. Note that both sides of (1.27) are random variables, the right-hand side through the starting value $X(s)$.

The function $(t, x, B) \mapsto P(t, x, B)$ is called the *transition function* of X. The value $P(t, x, B)$ is the probability of ending up in B if one started t periods ago in x. The transition function "almost" satisfies $P(0, x, B) = \varepsilon_0(B)$ and the **Chapman–Kolmogorov equation**

$$P(s+t, x, B) = \int P(t, y, B) P(s, x, dy) \tag{1.28}$$

for $s, t \geq 0$, $x \in E$, $B \in \mathscr{B}(E)$. Indeed, for arbitrary $r \geq 0$ we have

$$\begin{aligned} P(s+t, X(r), B) &= P(X(r+s+t) \in B | \mathscr{F}_r) \\ &= E\big(P(X(r+s+t) \in B | \mathscr{F}_{r+s}) \big| \mathscr{F}_r\big) \\ &= E\big(P(t, X(r+s), B) \big| \mathscr{F}_r\big) \\ &= \int P(t, y, B) P(s, X(r), dy), \end{aligned} \tag{1.29}$$

where the last line follows because $P(s, X(r), \cdot)$ is the law of $X(r+s)$ given the information up to time r. The restriction *almost* above refers to the fact that (1.29) yields (1.28) only up to some set of points x that is visited with probability 0 by X. To be more precise, we call $(P(t,x,\cdot))_{t\geq 0, x\in E}$ a **(homogeneous) transition function** only if (1.28) holds for any x without exception. Moreover, one generally requires the family in the above definition of a Markov process to be a transition function in this sense. Of course, it is unique only up to some set of points x that are never visited by the process.

In discrete time it suffices to consider one-step transitions in order to verify the Markov property:

Proposition 1.34 *Suppose that $P(X(t+1) \in B|\mathscr{F}_t) = Q(X(t), B)$, $t \geq 0$, $B \in \mathscr{B}(E)$ for some family $(Q(x,\cdot))_{x\in E}$ of probability measures on E. Then X is a Markov process relative to the transition function defined recursively by $P(0,x,B) = \varepsilon_x(B)$ and $P(t+1,x,B) = \int Q(y,B)P(t,x,dy)$.*

Proof Equation (1.27) follows by induction because

$$\begin{aligned} P(X(s+t+1) \in B|\mathscr{F}_s) &= E\big(P(X(s+t+1) \in B|\mathscr{F}_{s+t})\big|\mathscr{F}_s\big) \\ &= E\big(Q(X(s+t), B)\big|\mathscr{F}_s\big) \\ &= \int Q(y,B)P(t,X(s),dy) \\ &= P(t+1, X(s), B). \end{aligned}$$

Similarly, (1.28) is obtained by induction from

$$\begin{aligned} P(s+t+1,x,B) &= \int Q(y,B)P(s+t,x,dy) \\ &= \iint Q(y,B)P(t,z,dy)P(s,x,dz) \\ &= \int P(t+1,z,B)P(s,x,dz). \end{aligned}$$ □

Proposition 1.35 *An adapted process X is a Markov process if and only if it is of Markov type in the sense of Sect. 1.3.*

Proof Suppose that X is a Markov process. By (1.3) we have

$$\begin{aligned} K^X(t,B) &= E\big(1_B(X(t) - X(t-1))\big|\mathscr{F}_{t-1}\big) \\ &= \int 1_B(x - X(t-1))P^{X(t)|\mathscr{F}_{t-1}}(dx) \\ &= \int 1_B(x - X(t-1))P(1, X(t-1), dx), \end{aligned}$$

which is a function of $X(t-1)$ and B as claimed.

Conversely, assume that $K^X(t, B) = \kappa(X(t-1), B)$ with some function κ that is a probability measure in its second argument. Again by (1.3) we have

$$\begin{aligned} P(X(s+1) \in B|\mathscr{F}_s) &= E\big(1_B(X(s) + \Delta X(s+1))\big|\mathscr{F}_s\big) \\ &= \int 1_B(X(s)+x)P^{\Delta X(s+1)|\mathscr{F}_s}(dx) \\ &= \int 1_B(X(s)+x)\kappa(X(s), dx) \qquad (1.30) \\ &=: Q(X(s), B) \qquad (1.31) \end{aligned}$$

for a family $(Q(x, \cdot))_{x\in E}$ of probability measures on E. The Markov property follows from Proposition 1.34. □

Corollary 1.36 (Random Walks and Geometric Random Walks) *Random walks and geometric random walks are Markov processes.*

Proof In view of Proposition 1.35, this follows immediately from Proposition 1.24 and Example 1.25. □

Together with the law of $X(0)$, the transition function determines the distribution of the whole process uniquely, as the following formula shows.

Proposition 1.37 *For any $0 \le t_1 \le \cdots \le t_n$ and any bounded or nonnegative measurable function $f : E^n \to \mathbb{R}$, we have*

$$\begin{aligned} E(f(X(t_1), \dots, X(t_n))) = \int \cdots \int & f(x_1, \dots, x_n)P(t_n - t_{n-1}, x_{n-1}, dx_n) \\ & \cdots P(t_2 - t_1, x_1, dx_2)P(t_1, x_0, dx_1)P^{X(0)}(dx_0). \end{aligned}$$

Proof We proceed by induction on n. For $n = 1$ we have

$$\begin{aligned} E(f(X(t_1))) &= E\big(E(f(X(t_1))|\mathscr{F}_0)\big) \\ &= E\bigg(\int f(x_1)P^{X(t_1)|\mathscr{F}_0}(dx_1)\bigg) \\ &= E\bigg(\int f(x_1)P(t_1, X(0), dx_1)\bigg) \\ &= \iint f(x_1)P(t_1, x_0, dx_1)P^{X(0)}(dx_0), \end{aligned}$$

where the third equation follows from the Markov property (1.27). Suppose now that the assertion holds for $n-1$. Again by (1.27), we obtain

$$\begin{aligned} E(f(X(t_1),\dots,X(t_n))) &= E\big(E(f(X(t_1),\dots,X(t_n))|\mathscr{F}_{t_{n-1}})\big) \\ &= E\Big(\int f(X(t_1),\dots,X(t_{n-1}),x_n)P^{X(t_n)|\mathscr{F}_{t_{n-1}}}(dx_n)\Big) \\ &= E\Big(\int f(X(t_1),\dots,X(t_{n-1}),x_n)P(t_n-t_{n-1},X(t_{n-1}),dx_n)\Big) \\ &= E(g(X(t_1),\dots,X(t_{n-1}))) \end{aligned} \tag{1.32}$$

with

$$g(x_1,\dots,x_{n-1}) := \int f(x_1,\dots,x_{n-1},x_n)P(t_n-t_{n-1},x_{n-1},dx_n).$$

By assumption we have that (1.32) equals

$$\int\cdots\int g(x_1,\dots,x_{n-1})P(t_{n-1}-t_{n-2},x_{n-2},dx_{n-1}) \\ \cdots P(t_2-t_1,x_1,dx_2)P(t_1,x_0,dx_1)P^{X(0)}(dx_0),$$

which yields the claim. □

The integrator

$$P(t_n-t_{n-1},x_{n-1},dx_n)\cdots P(t_2-t_1,x_1,dx_2)P(t_1,x_0,dx_1)P^{X(0)}(dx_0)$$

stands for the joint law of $X(0), X(t_1), \dots, X(t_n)$. Specifically, the probability of observing $X(0), X(t_1), \dots, X(t_n)$ in $x_0, \dots, x_n$ is given by the probability of observing $X(0)$ in x_0, multiplied by the probability of moving from x_0 to x_1 in time t_1, multiplied by etc. up to the probability of moving from x_{n-1} to x_n in time $t_n - t_{n-1}$.

The transition function leads naturally to a family $(p_t)_{t\geq 0}$ of operators on bounded measurable functions $f: E \to \mathbb{R}$, called the **transition semigroup**. These operators are defined via

$$p_t f(x) := \int f(y)P(t,x,dy)$$

for $t \geq 0$, bounded measurable $f: E \to \mathbb{R}$, and $x \in E$. Put differently, $p_t f(x)$ is the expected value of $f(X(s+t))$ given that $X(s) = x$. In view of

$$p_t 1_B(x) = P(t,x,B),$$

the transition function can be recovered from $(p_t)_{t\geq 0}$. The family $(p_t)_{t\geq 0}$ is called *semigroup* because

$$p_{s+t} = p_s \circ p_t, \tag{1.33}$$

which means that $p_{s+t}f = p_s(p_t f)$ for any bounded measurable function f. Equation (1.33) follows easily from the Chapman–Kolmogorov equation.

The Chapman–Kolmogorov equation or Proposition 1.34 shows that in discrete time all p_t and hence the distribution of the whole process can be recovered from p_1. Alternatively, the transition probabilities can be derived from the **generator** G of the Markov process, which is another operator mapping bounded measurable functions $f : E \to \mathbb{R}$ on the like. It is defined as $Gf := p_1 - p_0$, i.e.

$$Gf(x) = p_1 f(x) - f(x). \tag{1.34}$$

In terms of the probability measures $Q(x, \cdot)$ in Proposition 1.34 or the function κ in (1.25), we can write it as

$$\begin{aligned} Gf(x) &= \int f(y)Q(x, dy) - f(x) \\ &= \int (f(x+y) - f(x))\kappa(x, dy), \end{aligned}$$

cf. (1.30, 1.31). For the random walk in Proposition 1.24 we have

$$Gf(x) = \int \big(f(x+y) - f(x)\big)\nu(dy).$$

The generator of the geometric random walk in Example 1.25 satisfies

$$Gf(x) = \int \big(f(xy) - f(x)\big)\varrho(dy).$$

Observe that the generator has the property that

$$\begin{aligned} M(t) &:= f(X(t)) - f(X(0)) - Gf(X_-) \bullet I \\ &= f(X(t)) - f(X(0)) - \sum_{s=1}^{t} Gf(X(s-1)) \end{aligned}$$

is a martingale for bounded measurable functions f because

$$\begin{aligned} E(M(t)|\mathscr{F}_{t-1}) - M(t-1) &= E\big(f(X(t)) - f(X(t-1)) - Gf(X(t-1))\big|\mathscr{F}_{t-1}\big) \\ &= p_1 f(X(t-1)) - f(X(t-1)) - Gf(X(t-1)) \\ &= 0. \end{aligned}$$

Moments $E(f(X(t)))$ of $X(t)$ can be computed by successive application of the generator G:

Proposition 1.38 (Backward Equation) *For bounded measurable $f : E \to \mathbb{R}$ the function $u(t,x) := p_t f(x)$ satisfies*

$$u(0,x) = f(x), \quad u(t,x) = u(t-1,x) + Gu(t-1,x),$$

where we use the notation

$$Gu(t-1,x) := (Gu(t-1,\cdot))(x). \tag{1.35}$$

Proof $u(0,x) = f(x)$ follows from $p_0 f = f$. By definition we have

$$\begin{aligned} u(t-1,x) + Gu(t-1,x) &= \int u(t-1,y) P(1,x,dy) \\ &= \iint f(z) P(t-1,y,dz) P(1,x,dy). \end{aligned}$$

The Chapman–Kolmogorov equation (1.28) yields

$$\iint f(z) P(t-1,y,dz) P(1,x,dy) = \int f(z) P(t,x,dz) \tag{1.36}$$

for $f = 1_B$. By standard arguments from measure theory, (1.36) actually holds for arbitrary bounded f. Indeed, both sides are linear in f and arbitrary measurable f can be approximated by linear combinations of indicator functions.

Since the right-hand-side of (1.36) is $p_t f(x) = u(t,x)$, we are done. □

In Mathematical Finance the previous proposition can, for instance, be applied to compute call option prices $E((X(T) - K)^+)$ for a stock following a geometric random walk.

Conversely, the law of $X(t)$ can be obtained from the law of $X(0)$ by successive application of the adjoint A of the generator G. To make this precise, denote by $B(E)$ the set of bounded measurable functions $E \to \mathbb{R}$. Any probability measure μ on E can be viewed as a linear mapping $B(E) \to \mathbb{R}$ via $\mu f := \int f d\mu$. We denote by $\mathscr{M}(E)$, $B(E)'$ the set of probability measures on E and the set of linear mappings $B(E) \to \mathbb{R}$, respectively. Moreover, we define the **adjoint operator** $A : \mathscr{M}(E) \to B(E)'$ of G by $(A\mu)f := \mu(Gf)$.

Proposition 1.39 (Forward Equation) *The laws $\mu_t := P^{X(t)}$, $t = 0, 1, 2, \ldots$ satisfy*

$$\mu_t - \mu_{t-1} = A\mu_{t-1}.$$

Proof We need to verify that $\mu_t f - \mu_{t-1} f = \mu_{t-1}(Gf)$ for any bounded measurable function $f : E \to \mathbb{R}$, i.e. $E(f(X(t))) - E(f(X(t-1))) = E(Gf(X(t-1)))$. Since $Gf(x) = p_1 f(x) - f(x)$ and $p_1 f(x) = \int f(y) P(1, x, dy)$, Proposition 1.37 yields

$$
\begin{aligned}
E(Gf(X(t-1))) &= E(p_1 f(X(t-1))) - E(f(X(t-1))) \\
&= \iiint f(x_2) P(1, x_1, dx_2) P(t-1, x_0, dx_1) P^{X(0)}(dx_0) - E(f(X(t-1))) \\
&= E(f(X(t))) - E(f(X(t-1)))
\end{aligned}
$$

and hence the claim. □

1.5 Optimal Control

In Mathematical Finance one often faces optimisation problems of various kinds, in particular when it comes to choosing trading strategies with in some sense maximal utility or minimal risk. Such problems can be tackled with different methods. We distinguish two approaches, which are discussed in the following sections and, for continuous time, in Chap. 7. As a motivation we first consider the simple situation of maximising a deterministic function of one or several variables.

Example 1.40

1. (Direct approach) Suppose that the goal is to maximise an *objective function*

$$
(x, \alpha) \mapsto \sum_{t=1}^{T} f(t, x_{t-1}, \alpha_t) + g(x_T) \tag{1.37}
$$

over all $x = (x_1, \ldots, x_T) \in (\mathbb{R}^d)^T$, $\alpha = (\alpha_1, \ldots, \alpha_T) \in A^T$ such that

$$
\Delta x_t := x_t - x_{t-1} = \delta(x_{t-1}, \alpha_t), \quad t = 1, \ldots, T
$$

for some given function $\delta : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$. The number or vector α_t stands for a dynamic *control* which determines the *state* x_t of the system. The reward (1.37) in turn depends primarily on x but possibly also on the control α itself. The initial value $x_0 \in \mathbb{R}^d$, the state space of controls $A \subset \mathbb{R}^m$ and the functions $f : \{1, \ldots, T\} \times \mathbb{R}^d \times A \to \mathbb{R}$, $g : \mathbb{R}^d \to \mathbb{R}$ are assumed to be given. The approach in Sect. 1.5.1 below corresponds to finding the maximum directly, without relying on smoothness or convexity of the functions f, g, δ or on topological properties of A. Rather, the idea is to reduce the problem to a sequence of simpler optimisations in just one A-valued variable α_t.

2. (Lagrange multiplier approach) Since the problem above concerns constrained optimisation, Lagrange multiplier techniques may make sense. To this end, define the *Lagrange function*

$$L(x, \alpha, y) := \sum_{t=1}^{T} f(t, x_{t-1}, \alpha_t) + g(x_T) - \sum_{t=1}^{T} y_t(\Delta x_t - \delta(x_{t-1}, \alpha_t))$$

on $(\mathbb{R}^d)^T \times A^T \times (\mathbb{R}^d)^T$. The usual first-order conditions lead us to look for a candidate $x^\star \in (\mathbb{R}^d)^T$, $\alpha^\star \in A^T$, $y^\star \in (\mathbb{R}^d)^T$ satisfying

a) $\Delta x_t^\star = \delta(x_{t-1}^\star, \alpha_t^\star)$ for $t = 1, \dots, T$, where we set $x_0^\star := x_0$,
b) $y_T^\star = \nabla g(x_T^\star)$,
c) $\Delta y_t^\star = -\nabla_x H(t, x_{t-1}^\star, \alpha_t^\star)$ for $t = 1, \dots, T$, where we set $H(t, \xi, a) := f(t, \xi, a) + y_t^\star \delta(\xi, a)$ and $\nabla_x H$ denotes the gradient of H viewed as a function of its second argument,
d) $\alpha_t^\star$ maximises $a \mapsto H(t, x_{t-1}^\star, a)$ on A for $t = 1, \dots, T$.

Provided that some convexity conditions hold, a)–d) are in fact sufficient for optimality of $\alpha^\star$:

Proposition 1.41 *Suppose that the set A is convex, $\xi \mapsto g(\xi)$, $(\xi, a) \mapsto H(t, \xi, a)$, $t = 1, \dots, T$ are concave and $\xi \mapsto g(\xi)$, $\xi \mapsto H(t, \xi, a)$, $t = 1, \dots, T$, $a \in A$ are differentiable. If conditions a)–d) hold, $(x^\star, \alpha^\star)$ is optimal for the problem in Example 1.40(1).*

Proof We set $h(t, \xi) := \sup_{a \in A} H(t, \xi, a)$ for any competitor (x, α) satisfying the constraints. Condition d) yields $h(t, x_{t-1}^\star) = H(t, x_{t-1}^\star, \alpha_t^\star)$ for $t = 1, \dots, T$. We have

$$\begin{aligned}
&\sum_{t=1}^{T} f(t, x_{t-1}, \alpha_t) + g(x_T) - \sum_{t=1}^{T} f(t, x_{t-1}^\star, \alpha_t^\star) - g(x_T^\star) \\
&\quad = \sum_{t=1}^{T} \Big(H(t, x_{t-1}, \alpha_t) - H(t, x_{t-1}^\star, \alpha_t^\star) - y_t^\star(\Delta x_t - \Delta x_t^\star) \Big) + g(x_T) - g(x_T^\star) \\
&\quad \leq \sum_{t=1}^{T} \Big((H(t, x_{t-1}, \alpha_t) - h(t, x_{t-1})) + \big(h(t, x_{t-1}) - h(t, x_{t-1}^\star) \big) \\
&\qquad - y_t^\star(\Delta x_t - \Delta x_t^\star) \Big) + \nabla g(x_T^\star)(x_T - x_T^\star) \\
&\quad \leq \sum_{t=1}^{T} \Big(\nabla_x h(t, x_{t-1}^\star)(x_{t-1} - x_{t-1}^\star) - y_t^\star(\Delta x_t - \Delta x_t^\star) \Big) \nabla g(x_T^\star)(x_T - x_T^\star)
\end{aligned} \tag{1.38}$$

$$= \sum_{t=1}^{T} \Big(- \Delta y_t^\star (x_{t-1} - x_{t-1}^\star) - y_t^\star (\Delta x_t - \Delta x_t^\star) \Big) + y_T^\star (x_T - x_T^\star) \tag{1.39}$$

$$= y_0^\star (x_0 - x_0^\star),$$

$$= 0$$

where the existence of $\nabla_x h(t, x_{t-1}^\star)$, inequality (1.38) as well as equation (1.39) follow from Proposition 1.42 below and the concavity of g. □

Under some more convexity (e.g. if δ is affine and $f(t, \cdot, \cdot)$ is concave for $t = 1, \dots, T$), the Lagrange multiplier solves some dual minimisation problem. This happens, for example, in the stochastic examples 1.71–1.76 in Sect. 1.5.4.

The following proposition is a version of the *envelope theorem* which concerns the derivative of the maximum of a parametrised function.

Proposition 1.42 *Let A be a convex set, $f : \mathbb{R}^d \times A \to \mathbb{R} \cup \{-\infty\}$ a concave function, and $\widetilde{f}(x) := \sup_{a \in A} f(x, a)$, $x \in \mathbb{R}^d$. Then $\widetilde{f}$ is concave. Suppose in addition that, for some fixed $x^\star \in \mathbb{R}^d$, the optimiser $a^\star := \arg\max_{a \in A} f(x^\star, a)$ exists and $x \mapsto f(x, a^\star)$ is differentiable in $x^\star$. Then $\widetilde{f}$ is differentiable in $x^\star$ with derivative*

$$D_i \widetilde{f}(x^\star) = D_i f(x^\star, a^\star), \quad i = 1, \dots, d. \tag{1.40}$$

Proof One easily verifies that $\widetilde{f}$ is concave. For $h \in \mathbb{R}^d$ we have

$$\begin{aligned} \widetilde{f}(x^\star + yh) &\geq f(x^\star + yh, a^\star) \\ &= f(x^\star, a^\star) + y \sum_{i=1}^{d} D_i f(x^\star, a^\star) h_i + o(y) \end{aligned} \tag{1.41}$$

as $y \in \mathbb{R}$ tends to 0. In view of [144, Proposition I.1.1.4],

$$s(y) := \frac{\widetilde{f}(x^\star + yh) - \widetilde{f}(x^\star)}{y}$$

is decreasing in $y \in \mathbb{R} \setminus \{0\}$. Denoting its limits in 0 by $s(0-)$, $s(0+)$, we obtain $s(0+) \leq s(0-) \leq \sum_{i=1}^{d} D_i f(x^\star, a^\star) h_i \leq s(0+)$ from (1.41). Consequently, $\widetilde{f}$ is differentiable in $x^\star$ with derivative (1.40). □

In the remainder of this section we discuss optimisation in a dynamic stochastic setup.

1.5.1 Dynamic Programming

Since we consider discrete-time stochastic control in this introductory chapter, we work on a filtered probability space $(\Omega, \mathscr{F}, \mathbf{F}, P)$ with filtration $\mathbf{F} = (\mathscr{F}_t)_{t=0,1,\dots,T}$ for finite T. For simplicity, we assume $\mathscr{F}_0$ to be trivial, i.e. all $\mathscr{F}_0$-measurable random variables are deterministic. By (1.1) this implies $E(X|\mathscr{F}_0) = E(X)$ for any random variable X.

Our goal is to maximise some expected reward $E(u(\alpha))$ over controls $\alpha \in \mathscr{A}$. The set $\mathscr{A}$ of **admissible controls** is a subset of all $\mathbb{R}^m$-valued adapted processes and it is assumed to be *stable under bifurcation*, i.e. for any stopping time τ, any event $F \in \mathscr{F}_\tau$, and any $\alpha, \widetilde{\alpha} \in \mathscr{A}$ with $\alpha^\tau = \widetilde{\alpha}^\tau$, the process $(\alpha|\tau_F|\widetilde{\alpha})$ defined by

$$(\alpha|\tau_F|\widetilde{\alpha})(t) := 1_{F^C}\alpha(t) + 1_F\widetilde{\alpha}(t)$$

is again an admissible control. Intuitively, this means that the decision how to continue may depend on the observations so far. Moreover, we suppose that $\alpha(0)$ coincides for all controls $\alpha \in \mathscr{A}$. The reward is expressed by some **reward function** $u : \Omega \times (\mathbb{R}^m)^{\{0,1,\dots,T\}} \to \mathbb{R} \cup \{-\infty\}$. For fixed $\alpha \in \mathscr{A}$, we use the shorthand $u(\alpha)$ for the random variable $\omega \mapsto u(\omega, \alpha(\omega))$. The reward is meant to refer to the final time $T \in \mathbb{N}$, which is expressed mathematically by the assumption that $u(\alpha)$ is $\mathscr{F}_T$-measurable for any $\alpha \in \mathscr{A}$.

Example 1.43 Typically, the reward function is of the form

$$u(\alpha) = \sum_{t=1}^{T} f(t, X^{(\alpha)}(t-1), \alpha(t)) + g(X^{(\alpha)}(T)) \tag{1.42}$$

for some functions $f : \{1, \dots, T\} \times \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R} \cup \{-\infty\}$, $g : \mathbb{R}^d \to \mathbb{R} \cup \{-\infty\}$, and $\mathbb{R}^d$-valued adapted *controlled processes* $X^{(\alpha)}$. The controlled process is assumed to depend on the control only up to the present time, i.e. for $t = 0, \dots, T$ we have $X^{(\alpha)}(t) = X^{(\widetilde{\alpha})}(t)$ if $\alpha^t = \widetilde{\alpha}^t$.

We call $\alpha^\star \in \mathscr{A}$ an **optimal control** if it maximises $E(u(\alpha))$ over all $\alpha \in \mathscr{A}$, where we set $E(u(\alpha)) := -\infty$ if $E(u(\alpha)^-) := -\infty$. Moreover, the **value process** of the optimisation problem is the family $(\mathscr{V}(\cdot, \alpha))_{\alpha \in \mathscr{A}}$ of adapted processes defined via

$$\mathscr{V}(t, \alpha) := \operatorname{ess\,sup}\left\{E(u(\widetilde{\alpha})|\mathscr{F}_t) : \widetilde{\alpha} \in \mathscr{A} \text{ with } \widetilde{\alpha}^t = \alpha^t\right\} \tag{1.43}$$

for $t = 0, \dots, T$ and $\alpha \in \mathscr{A}$. To this end, recall that the *essential supremum* in (1.43) is the smallest $\mathscr{F}_t$-random variable that dominates the right-hand side outside some null set, cf. Sect. A.2. The right-hand side of (1.43) represents the optimisation problem if we follow the control α up to time t and behave optimally afterwards.

The value process is characterised by some martingale/supermartingale property:

Theorem 1.44

1. *If* $\mathscr{V}(0) := \sup_{\alpha\in\mathscr{A}} E(u(\alpha)) \neq \pm\infty$*, the following holds.*
   a) $\mathscr{V}(t,\widetilde{\alpha}) = \mathscr{V}(t,\alpha)$ *if* $\widetilde{\alpha}^t = \alpha^t$.
   b) $(\mathscr{V}(t,\alpha))_{t\in\{0,\dots,T\}}$ *is a supermartingale with terminal value* $\mathscr{V}(T,\alpha) = u(\alpha)$ *for any admissible control* α *with* $E(u(\alpha)) > -\infty$.
   c) *If* $\alpha^\star$ *is an optimal control, then* $(\mathscr{V}(t,\alpha^\star))_{t\in\{0,\dots,T\}}$ *is a martingale.*
2. *Suppose that* $(\widetilde{\mathscr{V}}(\cdot,\alpha))_{\alpha\in\mathscr{A}}$ *is a family of processes such that*
   a) $\widetilde{\mathscr{V}}(t,\widetilde{\alpha}) = \widetilde{\mathscr{V}}(t,\alpha)$ *if* $\widetilde{\alpha}^t = \alpha^t$. *The common value* $\widetilde{\mathscr{V}}(0,\alpha)$ *is denoted by* $\widetilde{\mathscr{V}}(0)$,
   b) $(\widetilde{\mathscr{V}}(t,\alpha))_{t\in\{0,\dots,T\}}$ *is a supermartingale with terminal value* $\widetilde{\mathscr{V}}(T,\alpha) = u(\alpha)$ *for any admissible control* α *with* $E(u(\alpha)) > -\infty$,
   c) $(\widetilde{\mathscr{V}}(t,\alpha^\star))_{t\in\{0,\dots,T\}}$ *is a submartingale—and hence a martingale—for some admissible control* $\alpha^\star$ *with* $E(u(\alpha^\star)) > -\infty$.

   Then $\alpha^\star$ *is optimal and* $\mathscr{V}(t,\alpha^\star) = \widetilde{\mathscr{V}}(t,\alpha^\star)$ *for* $t = 0,\dots,T$. *In particular,* $\mathscr{V}(0) = \widetilde{\mathscr{V}}(0)$.

Proof

1. The first statement, adaptedness, and the terminal value of $\mathscr{V}(\cdot,\alpha)$ are evident. In order to show the supermartingale property, let $t \in \{0,\dots,T\}$. Stability under bifurcation implies that the set of all $E(u(\widetilde{\alpha})|\mathscr{F}_t)$ with $\widetilde{\alpha} \in \mathscr{A}$ satisfying $\widetilde{\alpha}^t = \alpha^t$ has the lattice property. By Lemma A.3 there exists a sequence of admissible controls α_n with $\alpha_n^t = \alpha^t$ and $E(u(\alpha_n)|\mathscr{F}_t) \uparrow \mathscr{V}(t,\alpha)$. For $s \leq t$ we have
$$E(E(u(\alpha_n)|\mathscr{F}_t)|\mathscr{F}_s) = E(u(\alpha_n)|\mathscr{F}_s) \leq \mathscr{V}(s,\alpha).$$
   The supermartingale property $E(\mathscr{V}(t,\alpha))|\mathscr{F}_s) \leq \mathscr{V}(s,\alpha)$ is now obtained by monotone convergence.
   Let $\alpha^\star$ be an optimal control. Since $\mathscr{V}(\cdot,\alpha^\star)$ is a supermartingale, the martingale property follows from
$$\mathscr{V}(0,\alpha^\star) = \sup_{\alpha\in\mathscr{A}} E(u(\alpha)) = E(u(\alpha^\star)) = E(\mathscr{V}(T,\alpha^\star))$$
   and Proposition 1.6.
2. The supermartingale property implies that
$$E(u(\alpha)) = E(\widetilde{\mathscr{V}}(T,\alpha)) \leq \widetilde{\mathscr{V}}(0,\alpha) = \widetilde{\mathscr{V}}(0) \tag{1.44}$$
   for any admissible control α. Since equality holds for $\alpha^\star$, we have that $\alpha^\star$ is optimal. By statement 1, $\mathscr{V}(\cdot,\alpha^\star)$ is a martingale with terminal value $u(\alpha^\star)$.

Since the same is true for $\widetilde{\mathscr{V}}(\cdot,\alpha^\star)$ by assumptions 2b,c), we have $\mathscr{V}(t,\alpha^\star) = E(u(\alpha^\star)|\mathscr{F}_t) = \widetilde{\mathscr{V}}(t,\alpha^\star)$ for $t = 0,\dots,T$. □

The previous theorem does not immediately lead to an optimal control but it often helps in order to verify that some candidate control is in fact optimal.

Remark 1.45

1. It may happen that the supremum in the definition of the optimal value is not a maximum, i.e. an optimal control does not exist. In this case Theorem 1.44 cannot be applied. Sometimes this problem can be circumvented by considering a certain closure of the set of admissible controls which does in fact contain the optimiser.
   If this is not feasible, a variation of Theorem 1.44(2) without assumption 2c) may be of interest. The supermartingale property 2b) of the candidate value process ensures that $\widetilde{\mathscr{V}}(0)$ is an upper bound of the optimal value $\mathscr{V}(0)$. If, for any $\varepsilon > 0$, one can find an admissible control $\alpha^{(\varepsilon)}$ with $\widetilde{\mathscr{V}}(0) \leq E(\widetilde{\mathscr{V}}(T,\alpha^{(\varepsilon)})) + \varepsilon$, then $\widetilde{\mathscr{V}}(0) = \mathscr{V}(0)$ and the $\alpha^{(\varepsilon)}$ yield a sequence of controls approaching this optimal value.
2. The conditions in statement 2 of Theorem 1.44 can also be used in order to obtain upper and lower bounds of the true value process $\mathscr{V}$. More specifically, if $(\widetilde{\mathscr{V}}(\cdot,\alpha))_{\alpha\in\mathscr{A}}$ denotes a family of processes such that 2a) and 2b) hold, then $\mathscr{V}(t,\alpha) \leq \widetilde{\mathscr{V}}(t,\alpha)$ for $t = 0,\dots,T$ and any admissible control α. We may even relax $\widetilde{\mathscr{V}}(T,\alpha) = u(\alpha)$ to $\widetilde{\mathscr{V}}(T,\alpha) \geq u(\alpha)$ in 2b).
   If, on the other hand, $(\widetilde{\mathscr{V}}(\cdot,\alpha))_{\alpha\in\mathscr{A}}$ denotes a family of processes such that 2a) and the submartingale property in 2c) hold for some control, then $\widetilde{\mathscr{V}}(0) \leq \mathscr{V}(0)$.
3. The above setup allows for a straightforward extension to the infinite time horizon $T = \infty$. However, one must be careful that *supermartingale* and *(sub-)martingale* in Theorem 1.44 refer to the time set $\{0,\dots,T\}$ as stated, i.e. including T. Specifically, $\mathscr{V}(\infty,\alpha)$ is defined and satisfies $E(\mathscr{V}(\infty,\alpha)|\mathscr{F}_t) \leq \mathscr{V}(t,\alpha)$ for $t \leq T$ etc.
4. In condition 2b) of the previous theorem it is enough to require that the process is a *local* supermartingale unless $T = \infty$, i.e. integrability need not be verified. Indeed, the inequality $E(u(\alpha)|\mathscr{F}_t) \leq \widetilde{\mathscr{V}}(t,\alpha)$ (1.44) is obtained recursively in time because we set $E(u(\alpha)) := -\infty$ if $E(u(\alpha)^-) := -\infty$.
5. Let us have a brief look at the intermediate optimisation problem (1.43) for later use. Fix $t \in \{0,\dots,T\}$. If $\widetilde{\mathscr{V}}(\cdot,\overline{\alpha}) - \widetilde{\mathscr{V}}(\cdot,\overline{\alpha})^t$ in Theorem 1.44(2) is a martingale for some admissible control $\overline{\alpha}$ with $E(u(\overline{\alpha})) > -\infty$, then $\mathscr{V}(s,\overline{\alpha}) = \widetilde{\mathscr{V}}(s,\overline{\alpha})$ and $\overline{\alpha}$ is the maximiser in the definition of $\mathscr{V}(s,\overline{\alpha})$ for $s = t,\dots,T$. This follows as in the proof of Theorem 1.44(2). In particular, the optimal control $\alpha^\star$ also maximises the conditional optimisation problem in the definition of $\mathscr{V}(t,\alpha^\star)$.

The value process can be obtained recursively starting from $t = T$:

Proposition 1.46 (Dynamic Programming Principle) *Suppose that $\mathscr{V}(0) \neq \pm\infty$. For $t = 1, \dots, T$ and any control $\alpha \in \mathscr{A}$ we have*

$$\mathscr{V}(t-1, \alpha) = \operatorname{ess\,sup}\left\{E(\mathscr{V}(t, \widetilde{\alpha})|\mathscr{F}_{t-1}) : \widetilde{\alpha} \in \mathscr{A} \textit{ with } \widetilde{\alpha}^{t-1} = \alpha^{t-1}\right\}.$$

Proof The inequality "$\leq$" is obvious because $E(u(\widetilde{\alpha})|\mathscr{F}_t) \leq \mathscr{V}(t, \widetilde{\alpha})$ and hence $E(u(\widetilde{\alpha})|\mathscr{F}_{t-1}) = E(E(u(\widetilde{\alpha})|\mathscr{F}_t)|\mathscr{F}_{t-1}) \leq E(\mathscr{V}(t, \widetilde{\alpha})|\mathscr{F}_{t-1})$.

In order to verify the converse inequality "$\geq$" fix a control $\widetilde{\alpha} \in \mathscr{A}$ with $\widetilde{\alpha}^{t-1} = \alpha^{t-1}$. Let $(\alpha_n)_{n=1,2,\dots}$ be a sequence of controls with $\alpha_n^t = \widetilde{\alpha}^t$ and such that $E(u(\alpha_n)|\mathscr{F}_t) \uparrow \mathscr{V}(t, \widetilde{\alpha})$ as $n \to \infty$. For any n we have

$$E(E(u(\alpha_n)|\mathscr{F}_t)|\mathscr{F}_{t-1}) = E(u(\alpha_n)|\mathscr{F}_{t-1}) \leq \mathscr{V}(t-1, \alpha).$$

Monotone convergence yields $E(\mathscr{V}(t, \widetilde{\alpha})|\mathscr{F}_{t-1}) \leq \mathscr{V}(t-1, \alpha)$, which implies the desired inequality. □

Moreover, the value process has a certain minimality property, which has already been mentioned in Remark 1.45(2):

Proposition 1.47 *Suppose that $\mathscr{V}(0) \neq \pm\infty$. If a family of processes $(\widetilde{\mathscr{V}}(\cdot, \alpha))_{\alpha \in \mathscr{A}}$ satisfies 2a,b) in Theorem 1.44, we have $\mathscr{V}(\cdot, \alpha) \leq \widetilde{\mathscr{V}}(\cdot, \alpha)$ for $t = 0, \dots, T$ and any control $\alpha \in \mathscr{A}$.*

Proof For any control $\widetilde{\alpha}$ satisfying $\widetilde{\alpha}^t = \alpha^t$ we have

$$E(u(\widetilde{\alpha})|\mathscr{F}_t) = E(\widetilde{\mathscr{V}}(T, \widetilde{\alpha})|\mathscr{F}_t) \leq \widetilde{\mathscr{V}}(t, \widetilde{\alpha}) = \widetilde{\mathscr{V}}(t, \alpha),$$

which implies that

$$\mathscr{V}(t, \alpha) = \operatorname{ess\,sup}\left\{E(u(\widetilde{\alpha})|\mathscr{F}_t) : \widetilde{\alpha} \in \mathscr{A} \text{ with } \widetilde{\alpha}^t = \alpha^t\right\} \leq \widetilde{\mathscr{V}}(t, \alpha). \qquad \square$$

As an example we consider the *Merton problem* of maximising the expected logarithmic utility of terminal wealth.

Example 1.48 (Logarithmic Utility of Terminal Wealth) An investor trades in a market consisting of a constant bank account and a stock whose price at time t equals

$$S(t) = S(0)\mathscr{E}(X)(t) = S(0)\prod_{s=1}^{t}(1 + \Delta X(s))$$

with $\Delta X(t) > -1$. Given that $\varphi(t)$ denotes the number of shares in the investor's portfolio from time $t-1$ to t, the profits from the stock investment in this period are $\varphi(t)\Delta S(t)$. If $v_0 > 0$ denotes the investor's initial endowment, her wealth at any time t amounts to

$$V_\varphi(t) := v_0 + \sum_{s=1}^{t} \varphi(s)\Delta S(s) = v_0 + \varphi \bullet S(t). \tag{1.45}$$

We assume that the investor's goal is to maximise the expected logarithmic utility $E(\log V_\varphi(T))$ of wealth at time T. To this end, we assume that the stock price process S is exogenously given and the investor's set of admissible controls is

$$\mathscr{A} := \{\varphi \text{ predictable} : V_\varphi > 0 \text{ and } \varphi(0) = 0\}.$$

It turns out that the problem becomes more transparent if we consider the *relative portfolio*

$$\pi(t) := \varphi(t)\frac{S(t-1)}{V_\varphi(t-1)}, \quad t = 1, \dots, T, \tag{1.46}$$

i.e. the fraction of wealth invested in the stock at time $t-1$. Starting with v_0, the stock holdings $\varphi(t)$ and the wealth process $V_\varphi(t)$ are recovered from π via

$$V_\varphi(t) = v_0 \mathscr{E}(\pi \bullet X)(t) = v_0 \prod_{s=1}^{t}(1 + \pi(s)\Delta X(s)) \tag{1.47}$$

and

$$\varphi(t) = \pi(t)\frac{V_\varphi(t-1)}{S(t-1)} = \pi(t)\frac{v_0 \mathscr{E}(\pi \bullet X)(t-1)}{S(t-1)}.$$

Indeed, (1.47) follows from

$$\Delta V_\varphi(t) = \varphi(t)\Delta S(t) = \frac{V_\varphi(t-1)\pi(t)}{S(t-1)}\Delta S(t) = V_\varphi(t-1)\Delta(\pi \bullet X)(t).$$

If $T = 1$, a simple calculation shows that the investor should buy $\varphi^\star(1) = \pi^\star(1)v_0/S(0)$ shares at time 0, where the optimal fraction $\pi^\star(1)$ maximises the function $\gamma \mapsto E(\log(1 + \gamma\Delta X(1)))$. We guess that the same essentially holds for multi-period markets, i.e. we assume that the optimal relative portfolio is obtained as the maximiser $\pi^\star(\omega, t)$ of the mapping

$$\gamma \mapsto E(\log(1 + \gamma\Delta X(t))|\mathscr{F}_{t-1})(\omega). \tag{1.48}$$

For simplicity we suppose that this maximiser exists and that $\log(1 + \pi^\star(t)\Delta X(t))$ has finite expectation for $t = 1, \dots, T$. Both can be shown to hold if the maximal achievable utility $\sup_{\varphi \in \mathscr{A}} E(\log V_\varphi(T))$ is finite, e.g. based on duality results in [126, 197].

The corresponding candidate value process is

$$
\begin{aligned}
\mathscr{V}(t,\varphi) &:= E\left(\log\left(V_\varphi(t)\prod_{s=t+1}^{T}\left(1+\pi^\star(s)\Delta X(s)\right)\right)\middle|\mathscr{F}_t\right)\\
&= \log V_\varphi(t) + E\left(\sum_{s=t+1}^{T}\log\left(1+\pi^\star(s)\Delta X(s)\right)\middle|\mathscr{F}_t\right),
\end{aligned}
\tag{1.49}
$$

where empty products are set to one and empty sums to zero as usual. Observe that

$$
\begin{aligned}
E(\mathscr{V}(t,\varphi)|\mathscr{F}_{t-1}) &= \log V_\varphi(t-1)\\
&\quad + E\left(\log\left(1+\frac{\varphi(t)S(t-1)}{V_\varphi(t-1)}\Delta X(t)\right)\middle|\mathscr{F}_{t-1}\right)\\
&\quad + E\left(\sum_{s=t+1}^{T}\log\left(1+\pi^\star(s)\Delta X(s)\right)\middle|\mathscr{F}_{t-1}\right)\\
&\le \log V_\varphi(t-1) + E\big(\log(1+\pi^\star(t)\Delta X(t))\big|\mathscr{F}_{t-1}\big)\\
&\quad + E\left(\sum_{s=t+1}^{T}\log\left(1+\pi^\star(s)\Delta X(s)\right)\middle|\mathscr{F}_{t-1}\right)\\
&= \mathscr{V}(t-1,\varphi)
\end{aligned}
$$

for any $t \geq 1$ and $\varphi \in \mathscr{A}$, with equality for the candidate optimiser $\varphi^\star$ satisfying $\varphi^\star(t) = \pi^\star(t)V_{\varphi^\star}(t-1)/S(t-1)$. By Theorem 1.44, we conclude that $\varphi^\star$ is indeed optimal.

Note that the optimiser or, more precisely, the optimal fraction of wealth invested in stock depends only on the local dynamics of the stock. This *myopic* property holds only for logarithmic utility.

The following variation of Example 1.48 considers utility of consumption rather than terminal wealth.

Example 1.49 (Logarithmic Utility of Consumption) In the market of the previous example the investor now spends $c(t)$ currency units at any time $t-1$. We assume that utility is derived from this consumption rather than terminal wealth, i.e. the goal is to maximise

$$
E\left(\sum_{t=1}^{T}\log c(t) + \log V_{\varphi,c}(T)\right) \tag{1.50}
$$

subject to the affordability constraint that the investor's wealth should stay positive:

$$0 < V_{\varphi,c}(t) := v_0 + \varphi \bullet S(t) - \sum_{s=1}^{t} c(s). \tag{1.51}$$

The last term $V_{\varphi,c}(T)$ in (1.50) refers to consumption of the remaining wealth at the end. The investor's set of admissible controls is

$$\mathscr{A} := \left\{(\varphi, c) \text{ predictable} : V_{\varphi,c} > 0, (\varphi, c)(0) = (0, 0)\right\}.$$

We try to come up with a reasonable candidate $(\varphi^\star, c^\star)$ for the optimal control. As in the previous example, matters simplify in relative terms. We write

$$\kappa(t) = \frac{c(t)}{V_{\varphi,c}(t-1)}, \quad t = 1, \dots, T \tag{1.52}$$

for the fraction of wealth that is consumed and

$$\pi(t) = \varphi(t)\frac{S(t-1)}{V_{\varphi,c}(t-1) - c(t)} = \varphi(t)\frac{S(t-1)}{V_{\varphi,c}(t-1)(1-\kappa(t))} \tag{1.53}$$

for the relative portfolio. Since the wealth after consumption at time $t-1$ is now $V_{\varphi,c}(t-1) - c(t)$, the numerator in (1.53) had to be adjusted. Similarly to (1.47), the wealth is given by

$$V_{\varphi,c}(t) = v_0 \prod_{s=1}^{t}(1-\kappa(s))(1+\pi(s)\Delta X(s)) = v_0 \mathscr{E}(-\kappa \bullet I)(t)\mathscr{E}(\pi \bullet X)(t). \tag{1.54}$$

We guess that the same relative portfolio as in the previous example is optimal in this modified setup, which leads to the candidate

$$\varphi^\star(t) = \pi^\star(t)\frac{V_{\varphi^\star,c^\star}(t-1) - c^\star(t)}{S(t-1)}.$$

As before we assume that $\pi^\star$ maximising (1.48) exists and that $\log(1+\pi^\star(t)\Delta X(t))$ has finite expectation for $t = 1, \dots, T$. Moreover, it may seem natural that the investor tries to spread consumption of wealth evenly over time. This idea leads to $\kappa^\star(t) = 1/(T+2-t)$ and hence

$$c^\star(t) = \frac{V_{\varphi^\star,c^\star}(t-1)}{T+2-t}$$

because at time $t-1$ there are $T+2-t$ periods left for consumption. This candidate pair $(\varphi^\star, c^\star)$ corresponds to the candidate value process

$$\begin{aligned}
\mathscr{V}(t,(\varphi,c)) &:= \sum_{s=1}^{t} \log c(s) \\
&\quad + E\Bigg(\sum_{s=t+1}^{T} \log\left(V_{\varphi,c}(t) \prod_{r=t+1}^{s-1} \Big((1-\kappa^\star(r))(1+\pi^\star(r)\Delta X(r))\Big)\kappa^\star(s)\right) \\
&\quad + \log\left(V_{\varphi,c}(t) \prod_{r=t+1}^{T} \Big((1-\kappa^\star(r))(1+\pi^\star(r)\Delta X(r))\Big)\right) \Bigg| \mathscr{F}_t \Bigg) \\
&= \sum_{s=1}^{t} \log c(s) + (T+1-t)\log V_{\varphi,c}(t) \\
&\quad + \sum_{r=t+1}^{T} \Bigg((T+1-r) E\Big(\log\big(1+\pi^\star(r)\Delta X(r)\big) \Big| \mathscr{F}_t \Big) \\
&\quad + (T+1-r)\log\tfrac{T+1-r}{T+2-r} - \log(T+2-r) \Bigg),
\end{aligned}$$

which is obtained if, starting from $t+1$, we invest the candidate fraction $\pi^\star$ of wealth in the stock and consume at any time $s-1$ the candidate fraction $\kappa^\star(s) = 1/(T+2-s)$ of wealth. In order to verify optimality, observe that

$$\begin{aligned}
E\big(\mathscr{V}(t,(\varphi,c))\big|\mathscr{F}_{t-1}\big) &= \sum_{s=1}^{t-1} \log c(s) + (T+1-t)\log V_{\varphi,c}(t-1) \\
&\quad + (T+1-t) E\Big(\log\big(1+\pi(t)\Delta X(t)\big) \Big| \mathscr{F}_{t-1} \Big) \\
&\quad + (T+1-t)\log(1-\kappa(t)) + \log\kappa(t) + \log V_{\varphi,c}(t-1) \\
&\quad + \sum_{s=t+1}^{T} \Bigg((T+1-s) E\Big(\log\big(1+\pi^\star(s)\Delta X(s)\big) \Big| \mathscr{F}_{t-1} \Big) \\
&\quad + (T+1-s)\log\tfrac{T+1-s}{T+2-s} - \log(T+2-s) \Bigg) \\
&\le \sum_{s=1}^{t-1} \log c(s) + (T+2-t)\log V_{\varphi,c}(t-1) \\
&\quad + (T+1-t) E\Big(\log\big(1+\pi^\star(t)\Delta X(t)\big) \Big| \mathscr{F}_{t-1} \Big)
\end{aligned}$$

$$
\begin{aligned}
&+ (T+1-t)\log(1-\tfrac{1}{T+2-t}) - \log(T+2-t) \\
&+ \sum_{s=t+1}^{T} \Big((T+1-s) E\Big(\log\big(1+\pi^\star(s)\Delta X(s)\big) \Big| \mathscr{F}_{t-1}\Big) \\
&+ (T+1-s)\log\tfrac{T+1-s}{T+2-s} - \log(T+2-s)\Big) \\
= {}& \mathscr{V}(t-1,(\varphi,c))
\end{aligned}
$$

for any admissible control (φ, c), where $\pi(t), \kappa(t)$ are defined as in (1.53, 1.52). To wit, the inequality holds because $\pi^\star(t)$ maximises $\gamma \mapsto E(\log(1+\gamma\Delta X(t))|\mathscr{F}_{t-1})$ and $1/(T+2-t)$ maximises $\delta \mapsto (T+1-t)\log(1-\delta)+\log\delta$. Again, equality holds if $(\varphi, c) = (\varphi^\star, c^\star)$. By Theorem 1.44 we conclude that $(\varphi^\star, c^\star)$ is indeed optimal.

The optimal consumption rate changes slightly if the objective is to maximise $E(\sum_{t=1}^{T} e^{-\delta(t-1)}\log c(t) + e^{-\delta T}\log V_{\varphi,c}(T))$ with some *impatience rate* $\delta \geq 0$.

The two previous examples allow for a straightforward extension to $d > 1$ assets. We continue with a second example which is also motivated by Mathematical Finance.

Example 1.50 (Quadratic Hedging) In the context of option hedging, the question arises how to approximate a given random variable by the terminal value of a stochastic integral relative to a given process. The random variable represents the payoff of a contingent obligation and the stochastic integral stands for the profits of an investor as in Example 1.48, cf. Chap. 12 for details.

More specifically, let X denote a square-integrable random variable, v_0 a real number, and S a martingale with $E(S(t)^2) < \infty$ for $t = 0, \dots, T$. The aim is to minimise the so-called *expected squared hedging error*

$$
E\big((V_\varphi(T) - X)^2\big) \tag{1.55}
$$

where $V_\varphi(t) := v_0 + \varphi \bullet S(t)$ represents the wealth of the investor except for the obligation X and φ ranges over all admissible controls in the set

$$
\begin{aligned}
\mathscr{A} &:= \{\varphi \text{ predictable} : \varphi(0) = 0 \text{ and } E((V_\varphi(T) - X)^2) < \infty\} \\
&= \{\varphi \text{ predictable} : \varphi(0) = 0 \text{ and } E(V_\varphi(T)^2) < \infty\} \\
&= \{\varphi \text{ predictable} : \varphi(0) = 0 \text{ and } E(\varphi^2 \bullet \langle S, S\rangle(T)) < \infty\}.
\end{aligned}
$$

The problem becomes simpler by introducing the martingale $V(t) := E(X|\mathscr{F}_t)$ generated by X and also the martingale $M_\varphi := V_\varphi - V$, which means that we have to maximise $E(u(\varphi))$ with $u(\varphi) := -M_\varphi(T)^2$. Integration by parts yields

$$\begin{aligned}M_\varphi(t)^2 &= M_\varphi(0)^2 + 2(M_\varphi)_- \bullet M_\varphi(t) + [M_\varphi, M_\varphi](t)\\ &= M_\varphi(0)^2 + N(t) + \langle M_\varphi, M_\varphi\rangle(t),\end{aligned}$$

where $N := 2(M_\varphi)_- \bullet M + [M_\varphi, M_\varphi] - \langle M_\varphi, M_\varphi\rangle$ is a martingale by Propositions 1.14(7) and 1.8. Consequently,

$$E(u(\varphi)|\mathscr{F}_t) = -M_\varphi(t)^2 - E\big(\langle M_\varphi, M_\varphi\rangle(T) - \langle M_\varphi, M_\varphi\rangle(t)\big|\mathscr{F}_t\big).$$

This expression is to be minimised in the definition (1.43) of the value process. Observe that $M_\varphi(t)$ only depends on φ^t. Moreover,

$$\begin{aligned}\langle M_\varphi, M_\varphi\rangle(T) - \langle M_\varphi, M_\varphi\rangle(t) &= \sum_{s=t+1}^{T} \Delta\langle M_\varphi, M_\varphi\rangle(s)\\ &= \sum_{s=t+1}^{T} \Big(\varphi(s)^2\Delta\langle S, S\rangle(s) - 2\varphi(s)\Delta\langle S, V\rangle(s) + \Delta\langle V, V\rangle(s)\Big)\end{aligned}$$

can be optimised separately for any s. It is easy to see that

$$\varphi(s) \mapsto \varphi(s)^2\Delta\langle S, S\rangle(s) - 2\varphi(s)\Delta\langle S, V\rangle(s) + \Delta\langle V, V\rangle(s) \tag{1.56}$$

is minimised by

$$\varphi^\star(s) := \frac{\Delta\langle S, V\rangle(s)}{\Delta\langle S, S\rangle(s)} \tag{1.57}$$

with minimal value

$$\Delta\langle V, V\rangle(s) - \varphi^\star(s)^2\Delta\langle S, S\rangle(s).$$

This leads to the ansatz

$$\mathscr{V}(t, \varphi) := -M_\varphi(t)^2 - \sum_{s=t+1}^{T} E\left(\Delta\langle V, V\rangle(s) - \varphi^\star(s)^2\Delta\langle S, S\rangle(s)\middle|\mathscr{F}_t\right) \tag{1.58}$$

for the value process, with $\varphi^\star$ defined in (1.57). Since $0 \leq \varphi^\star(s)^2\Delta\langle S, S\rangle(s) \leq \Delta\langle V, V\rangle(s)$, the control $\varphi^\star$ is admissible. Obviously, we have $\mathscr{V}(T, \varphi) = u(\varphi)$. For $t = 1, \dots, T$ we obtain

$$\begin{aligned}
E(\mathscr{V}(t, \varphi)|\mathscr{F}_{t-1}) &= \mathscr{V}(t-1, \varphi) - E\big(M_\varphi(t)^2 - M_\varphi(t-1)^2\big|\mathscr{F}_{t-1}\big) \\
&\quad + \Delta\langle V, V\rangle(t) - \varphi^\star(t)^2\Delta\langle S, S\rangle(t) \\
&= \mathscr{V}(t-1, \varphi) - \big(\varphi(t)^2\Delta\langle S, S\rangle(t) - 2\varphi(t)\Delta\langle S, V\rangle(t) + \Delta\langle V, V\rangle(t)\big) \\
&\quad + \varphi^\star(t)^2\Delta\langle S, S\rangle(t) - 2\varphi^\star(t)\Delta\langle S, V\rangle(t) + \Delta\langle V, V\rangle(t).
\end{aligned}$$

Since (1.56) is minimised by $\varphi^\star(s)$, we have $E(\mathscr{V}(t, \varphi)|\mathscr{F}_{t-1}) \leq \mathscr{V}(t-1, \varphi)$ with equality for $\varphi = \varphi^\star$. Theorem 1.44 yields that $\varphi^\star$ is indeed optimal. Moreover, the optimal value of the original control problem (1.55) amounts to

$$\begin{aligned}
-\mathscr{V}(0, \varphi^\star) &= (V_{\varphi^\star}(0) - V(0))^2 + \sum_{t=1}^{T} E\Big(\Delta\langle V, V\rangle(t) - \varphi^\star(t)^2\Delta\langle S, S\rangle(t)\Big) \\
&= (v_0 - E(X))^2 + E\Big(\langle V, V\rangle(T) - (\varphi^\star)^2 \bullet \langle S, S\rangle(T)\Big).
\end{aligned}$$

If this is to be minimised over the investor's initial capital v_0 as well, the optimal choice is obviously $v_0 = E(X)$, leading to the optimal value

$$E\Big(\langle V, V\rangle(T) - (\varphi^\star)^2 \bullet \langle S, S\rangle(T)\Big)$$

of the control problem.

1.5.2 Optimal Stopping

An important subclass of control problems concerns *optimal stopping*. Given some time horizon $T < \infty$ and some adapted process X with $E(\sup_{t\in\{0,\dots,T\}} |X(t)|) < \infty$, the goal is to maximise the *expected reward*

$$\tau \mapsto E(X(\tau)) \tag{1.59}$$

over all stopping times τ with values in $\{0, \dots, T\}$.

Remark 1.51

1. In the spirit of Sect. 1.5.1, a stopping time τ can be identified with the corresponding adapted process $\alpha(t) := 1_{\{t\leq\tau\}}$, and hence $X(\tau) = X(0) + \alpha \bullet X(T)$. Put differently, $\mathscr{A} := \{\alpha$ predictable: α $\{0, 1\}$-valued, decreasing, $\alpha(0) = 1\}$ and $u(\alpha) := X(0) + \alpha \bullet X(T)$ in Sect. 1.5.1 lead to the above optimal stopping problem.
2. Sometimes X may not be adapted in applications such as in Example 1.58 below. Then we can replace it with the adapted process $\widetilde{X}(t) := E(X(t)|\mathscr{F}_t)$. Indeed, we have

$$\begin{aligned} E(X(\tau)) &= \sum_{t=0}^{T} E\big(X(t)1_{\{\tau=t\}}\big) \\ &= \sum_{t=0}^{T} E\Big(E\big(X(t)\big|\mathscr{F}_t\big)1_{\{\tau=t\}}\Big) \\ &= \sum_{t=0}^{T} E\big(\widetilde{X}(t)1_{\{\tau=t\}}\big) \\ &= E(\widetilde{X}(\tau)) \end{aligned}$$

because $\{\tau = t\} \in \mathscr{F}_t$ for any stopping time τ.

The role of the value process in general control problems is now taken by the **Snell envelope** of X, which denotes the adapted process V given by

$$V(t) := \operatorname{ess\,sup}\big\{E(X(\tau)|\mathscr{F}_t) : \tau \text{ stopping time with values in } \{t, t+1, \dots, T\}\big\}. \tag{1.60}$$

It represents the maximal expected reward if we start at time t and have not stopped yet. The following martingale criterion may be helpful to verify the optimality of a candidate stopping time. We will apply its continuous-time version in an example in Chap. 7.

Proposition 1.52

1. *Let τ be a stopping time with values in $\{0, \dots, T\}$. If V is an adapted process such that V^τ is a martingale, $V(\tau) = X(\tau)$, and $V(0) = M(0)$ for some martingale (or at least supermartingale) $M \geq X$, then τ is optimal for (1.59) and V coincides up to time τ with the Snell envelope of X.*
2. *More generally, let τ be a stopping time with values in $\{t, \dots, T\}$ and $F \in \mathscr{F}_t$. Suppose that M is a process with $M \geq X$ on F and such that $M - M^t$ is a martingale (or at least a supermartingale). If V is an adapted process such that*

$V^\tau - V^t$ is a martingale, $V(\tau) = X(\tau)$, and $V(t) = M(t)$ on F, then $V(s)$ coincides on F for $t \le s \le \tau$ with the Snell envelope of X and τ is optimal on F for (1.60), i.e. it maximises $E(X(\tau)|\mathscr{F}_t)$ on F.

Proof

1. This follows from the second statement.
2. Let $s \in \{t, \dots, T\}$. We have to show that

$$V(s) = \operatorname{ess\,sup}\left\{E(X(\widetilde{\tau})|\mathscr{F}_s) : \widetilde{\tau} \text{ stopping time with values in } \{s, \dots, T\}\right\}$$

holds on $\{s \le \tau\} \cap F$.
"$\le$": On $\{s \le \tau\} \cap F$ we have

$$\begin{aligned} V(s) &= E(V^\tau(T)|\mathscr{F}_s) \\ &= E(X(\tau \vee s)|\mathscr{F}_s) \\ &\le \operatorname{ess\,sup}\left\{E(X(\widetilde{\tau})|\mathscr{F}_s) : \widetilde{\tau} \text{ stopping time with values in } \{s, \dots, T\}\right\}. \end{aligned} \tag{1.61}$$

"$\ge$": Note that $V^\tau(T) = V(\tau) = X(\tau) \le M(\tau) = M^\tau(T)$ and Lemma 1.7 yield

$$V^\tau(s) = M^\tau(s) \ge E(M(\widetilde{\tau})|\mathscr{F}_s) \ge E(X(\widetilde{\tau})|\mathscr{F}_s) \tag{1.62}$$

on $\{s \le \tau\} \cap F$ for any stopping time $\widetilde{\tau} \ge s$.
(1.61) and (1.62) yield that τ is optimal on F. Indeed, $E(X(\tau)|\mathscr{F}_t) = V(t) \ge E(X(\widetilde{\tau})|\mathscr{F}_t)$ holds for any stopping time $\widetilde{\tau} \ge t$. □

The following more common verification result corresponds to Theorem 1.44 in the framework of optimal stopping.

Theorem 1.53

1. *Let τ be a stopping time with values in $\{0, \dots, T\}$. If $V \ge X$ is a supermartingale such that V^τ is a martingale and $V(\tau) = X(\tau)$, then τ is optimal for (1.59) and V coincides up to time τ with the Snell envelope of X.*
2. *More generally, let τ be a stopping time with values in $\{t, \dots, T\}$. If $V \ge X$ is an adapted process such that $V - V^t$ is a supermartingale, $V^\tau - V^t$ is a martingale, and $V(\tau) = X(\tau)$, then $V(s)$ coincides for $t \le s \le \tau$ with the Snell envelope of X and τ is optimal for (1.60), i.e. it maximises $E(X(\tau)|\mathscr{F}_t)$.*
3. *If τ is optimal for (1.60) and V denotes the Snell envelope, they have the properties in statement 2.*

Proof

1. This follows from the second statement.
2. This immediately follows from choosing $M = V$ and $F = \Omega$ in statement 2 of Proposition 1.52 but we give a short direct proof as well. For any competing stopping time $\widetilde{\tau}$ with values in $\{t, t+1, \dots, T\}$, Proposition 1.9 yields

$$E(X(\widetilde{\tau})|\mathscr{F}_t) \leq E(V(\widetilde{\tau})|\mathscr{F}_t) = E(V^{\widetilde{\tau}}(T)|\mathscr{F}_t) \leq V^{\widetilde{\tau}}(t) = V(t),$$

with equality everywhere for $\widetilde{\tau} = \tau$.
3. $\tau \geq t$, $V \geq X$, and adaptedness of V are obvious. It remains to be shown that $V - V^t$ is a supermartingale, $V^\tau - V^t$ is a martingale, and $V(\tau) = X(\tau)$. Using the identification of Remark 1.51(1), we have $V(t) = \mathscr{V}(t, 1)$. Theorem 1.44(1) yields that V and hence also $V - V^t$ are supermartingales. In particular, $V(t) \geq E(V(\tau)|\mathscr{F}_t)$. But optimality of τ implies

$$V(t) = E(X(\tau)|\mathscr{F}_t) \leq E(V(\tau)|\mathscr{F}_t). \tag{1.63}$$

Hence, equality holds in (1.63), which yields $X(\tau) = V(\tau)$ and $E(V^\tau(T) - V^t(T)) = 0 = V^\tau(0) - V^t(0)$. The martingale property of $V^\tau - V^t$ now follows from Proposition 1.6. □

Remark 1.54 Statement 3 in Theorem 1.53 shows that the sufficient condition in Proposition 1.52(1) is necessary, i.e. for the Snell envelope V there exists a martingale M as in statement 1 of this proposition. Indeed, one may choose $M := V(0) + M^V$ if $V = V(0) + M^V + A^V$ denotes the Doob decomposition of V.

Hence, Proposition 1.52 can in principle be used to determine the Snell envelope numerically, namely by minimising $M(0)$ over all martingales dominating X. The resulting process coincides with the Snell envelope up to time τ. Cf. also Remark 7.20 in this context.

The following result helps to determine both the Snell envelope and an optimal stopping time. The first statement corresponds to the backward recursion of Proposition 1.46 in the context of optimal stopping, the third to Proposition 1.47.

Proposition 1.55 *Let V denote the Snell envelope.*

1. *V is obtained recursively by $V(T) = X(T)$ and*

$$V(t-1) = \max\left\{X(t-1), E(V(t)|\mathscr{F}_{t-1})\right\}, \quad t = T-1, \dots, 0. \tag{1.64}$$

2. *The stopping times*

$$\underline{\tau}_t := \inf\left\{s \in \{t, \dots, T\} : V(s) = X(s)\right\}$$

and

$$\overline{\tau}_t := T \wedge \inf\left\{s \in \{t, \dots, T-1\} : E(V(s+1)|\mathscr{F}_s) < X(s)\right\} \tag{1.65}$$

are optimal in (1.60), i.e. they maximise $E(X(\tau)|\mathscr{F}_t)$.

3. *V is the smallest supermartingale dominating X.*

Proof

1. and 2. Define V recursively as in statement 1 rather than as in (1.60) and set $\tau = \underline{\tau}_t$ or $\tau = \overline{\tau}_t$. We show that τ and V satisfy the conditions of Theorem 1.53(2). It is obvious that V is a supermartingale with $V \geq X$ and $V(\tau) = X(\tau)$. Fix $s \in \{t+1, t+2, \dots, T\}$. On the set $\{s \leq \tau\}$ we have $V(s-1) = E(V(s)|\mathscr{F}_{s-1})$ by definition of τ. Hence $V^\tau - V^t$ is a martingale.
3. In view of statement 1 or Theorem 1.53(3) it remains to be shown that $V \leq W$ for any supermartingale dominating X. To this end observe that

$$\begin{aligned} V(t) &= \operatorname{ess\,sup}\left\{E(X(\tau)|\mathscr{F}_t) : \tau \text{ stopping time in } \{t, t+1, \dots, T\}\right\} \\ &\leq \operatorname{ess\,sup}\left\{E(W(\tau)|\mathscr{F}_t) : \tau \text{ stopping time in } \{t, t+1, \dots, T\}\right\} \\ &\leq W(t) \end{aligned}$$

by Proposition 1.9. □

In some sense, $\underline{\tau}_t$ is the earliest and $\overline{\tau}_t$ the latest optimal stopping time for (1.60). Often the optimal stopping time is unique, in which case we have $\underline{\tau}_t = \overline{\tau}_t$. The solution to the original stopping problem (1.59) is obtained for $t = 0$. In particular, Proposition 1.55(2) yields that an optimiser of (1.59) exists.

Remark 1.56

1. In terms of the drift coefficient a^V of V as in Proposition 1.31, (1.64) and (1.65) can be reformulated as

$$\max\left\{a^V(t), X(t-1) - V(t-1)\right\} = 0, \quad t = 1, \dots, T \tag{1.66}$$

and

$$\begin{aligned} \overline{\tau}(t) &= T \wedge \inf\left\{s \in \{t, \dots T-1\} : a^V(t+1) < X(t) - V(t)\right\} \\ &= T \wedge \inf\left\{s \in \{t, \dots T-1\} : a^V(t+1) < 0\right\}, \end{aligned} \tag{1.67}$$

respectively. The second equality in (1.67) follows from (1.66).
2. If the reward process X is nonnegative, one may also consider the infinite time horizon $T = \infty$, in which case we set $X(\infty) := 0$. Propositions 1.52, 1.55(1,3) and Theorem 1.53 remain true in this case if we make sure that the martingale resp. supermartingale property always refers to the time set $\{0, \dots, T\}$ including

$T = \infty$, cf. Remark 1.45(3). However, it is not obvious how to obtain V from the "infinite recursion" (1.64) any more. Moreover, an optimal stopping time may fail to exist.

As an example we consider the price of a perpetual put option in the so-called Cox–Ross–Rubinstein model.

Example 1.57 (Perpetual Put) Suppose that $S(t)/S(0)$ is a geometric random walk such that $S(t)/S(t-1)$ has only two possible values $u > 1, d := 1/u < 1$ which are assumed with probabilities p resp. $1-p$. In particular, the process S moves only on the grid $S(0)u^{\mathbb{Z}}$. Consider the reward process $X(t) = e^{-rt}(K - S(t))^+$, $t = 0, 1, 2, \dots$ with some constants $r > 0$, $K > 0$. In the context of Mathematical Finance, $S(t)$ represents a stock price and $X(t)$ the discounted payoff of a perpetual put option on the stock that is exercised at time t. The goal is to maximise the expected discounted payoff $E(X(\tau))$ over all stopping times τ.

We make the natural ansatz that the Snell envelope is of the form

$$V(t) = e^{-rt}v(S(t)) \tag{1.68}$$

for some function v. If we guess that it is optimal to stop when $S(t)$ falls below some threshold $s^\star \leq K$, we must have $v(s) = (K-s)^+$ for $s \leq s^\star$. As long as $S(t) > s^\star$, the process $e^{-rt}v(S(t))$ should behave as a martingale, i.e.

$$\begin{aligned} 0 &= E(\Delta V(t)|\mathscr{F}_{t-1}) \\ &= e^{-rt}\big(pv(S(t-1)u) + (1-p)v(S(t-1)d) - e^r v(S(t-1))\big), \end{aligned} \tag{1.69}$$

which only holds if v grows in the right way for $s > s^\star$. A power ansatz for v turns out to be successful, i.e.

$$v(s) = \begin{cases} K - s & \text{for } s \leq s^\star, \\ cs^{-a} & \text{for } s > s^\star \end{cases}$$

with some constants $s^\star, a, c > 0$. In order for (1.69) to hold, a needs to be chosen such that $pu^{-a} + (1-p)u^a = e^r$, i.e.

$$u^{-a} = \frac{e^r - \sqrt{e^{2r} - 4p(1-p)}}{2p}.$$

Subsequently, we hope that some sort of contact condition at the boundary leads to the solution, which is often true for optimal stopping problems. More specifically, we choose the largest $c > 0$ such that the linear function $s \mapsto K - s$ and the decreasing convex function $s \mapsto cs^{-a}$ coincide at least in one element $s^\star$ in $S(0)u^{\mathbb{Z}}$, i.e. $s^\star = S(0)u^k$ for some integer k. Define the stopping time $\tau := \inf\{t \geq 0 : S(t) \leq s^\star\}$ and V as in (1.68) with $V(\infty) = 0$. We suppose that $p \leq 1/2$ so that τ is almost surely finite. It is now easy to verify that $V \geq X$, $V(\tau) = X(\tau)$, and V^τ is a martingale. By dominated convergence, the martingale property actually holds on $\{0, \dots, \infty\} = \mathbb{N} \cup \{\infty\}$ rather than just $\mathbb{N}$. In view of Theorem 1.53, it remains to show that V is a supermartingale. To this end, fix $t \in \{1, \dots, T\}$. On $\{S(t-1) > s^\star\}$ we have

$$E(V(t)|\mathscr{F}_{t-1}) = E(e^{-rt} cS(t)^{-a}|\mathscr{F}_{t-1}) = e^{-r(t-1)} cS(t-1)^{-a} = V(t-1).$$

Similarly, we have

$$E(V(t)|\mathscr{F}_{t-1}) \leq E(e^{-r(t)} cS(t)^{-a}|\mathscr{F}_{t-1}) = e^{-r(t-1)} cS(t-1)^{-a} = V(t-1)$$

on $\{S(t-1) = s^\star\}$. On $\{S(t-1) < s^\star\}$ we argue that

$$\begin{aligned}
E(V(t)|\mathscr{F}_{t-1}) &= E\big(e^{-rt}(K - S(t))\big|\mathscr{F}_{t-1}\big) \\
&= e^{-rt} K\left(1 - \frac{S(t-1)}{s^\star}\right) + \frac{S(t-1)}{s^\star} E\left(e^{-rt}\left(K - \frac{S(t)}{S(t-1)} s^\star\right)\middle|\mathscr{F}_{t-1}\right) \\
&\leq e^{-rt} K\left(1 - \frac{S(t-1)}{s^\star}\right) + \frac{S(t-1)}{s^\star} E\left(e^{-rt} c\left(\frac{S(t)}{S(t-1)} s^\star\right)^{-a}\middle|\mathscr{F}_{t-1}\right) \\
&= e^{-rt} K\left(1 - \frac{S(t-1)}{s^\star}\right) + \frac{S(t-1)}{s^\star} e^{-r(t-1)} c(s^\star)^{-a} \\
&= e^{-rt} K\left(1 - \frac{S(t-1)}{s^\star}\right) + \frac{S(t-1)}{s^\star} e^{-r(t-1)} (K - s^\star) \\
&\leq e^{-r(t-1)}(K - S(t-1)) = V(t-1),
\end{aligned}$$

which finishes the proof.

If u is small, we can compute approximations of $c, s^\star$. Indeed, at $s^\star$ the linear function $s \mapsto K - s$ is almost tangent to $s \mapsto cs^{-a}$, i.e. their derivatives coincide. The two conditions $cs^{\star -a} = K - s^\star$ and $-acs^{\star -a-1} \approx -1$ are solved by $s^\star \approx Ka/(1+a)$ and $c \approx K^{1+a} a^a/(1+a)^{1+a}$.

As another example we consider the famous so-called *secretary* or *marriage problem*.

Example 1.58 (Marriage Problem) Suppose that you are looking for a spouse, a flat, or an employee. We assume that time permits you to examine n candidates before a decision must be made. Moreover, you are supposed to be satisfied only with the best of all T aspirants. The problem is that you can inspect them only one by one and that you have to opt for or against any candidate before you can see the next one. The goal is to maximise the probability of choosing the best one.

In mathematical terms we assume the ranks $1, \dots, T$ of the candidates to appear in totally random order. The rank of candidate t is denoted by $N(t)$. But when applicant t shows up, you only observe his or her *relative rank* $R(t)$ compared to all previous applicants. Since any permutation is assumed to be equally likely, it is not hard to show that the law of $R(t)$ is uniform on $1, \ldots, t$, independently of the past, i.e. $R(t)$ is independent of $R(1), \ldots, R(t-1)$. The decision to stop must be based on the observed relative ranks, i.e. on the filtration $\mathscr{F}_t := \sigma(R(1), \ldots, R(t))$. The goal is to maximise $P(N(\tau) = 1)$ among all stopping times τ with values $1, 2, \ldots, T$. In order to make this problem look as in (1.59), note that $P(N(\tau) = 1) = E(X(\tau))$ for $X(t) := 1_{\{N(t)=1\}}$. However, the reward process X is not adapted to our filtration. In line with Remark 1.51(2), we replace it with its conditional expectation $\widetilde{X}(t) = E(X(t)|\mathscr{F}_t)$. It is easy to see that

$$\widetilde{X}(t) = \begin{cases} 0 & \text{if } R(t) = 0, \\ t/T & \text{if } R(t) = 1. \end{cases}$$

Indeed, t cannot be globally optimal if not even the relative rank is 1. If, on the other hand, it has relative rank $R(t) = 1$, it is globally optimal if and only if the global optimiser is among the first t candidates. This happens with probability t/T, independently of the observed relative ranks $R(1), \ldots, R(t)$.

We can now determine the Snell envelope recursively according to $V(T) = \widetilde{X}(T)$ and (1.64) with $\widetilde{X}$ instead of X. We obtain $V(T) = 1_{\{R(T)=1\}}$ and

$$\begin{aligned} V(T-1) &= \max\left\{\widetilde{X}(T-1), E(V(T)|\mathscr{F}_{T-1})\right\} \\ &= \max\left\{\frac{T-1}{T} 1_{\{R(T-1)=1\}}, P(R(T) = 1)\right\} \\ &= \max\left\{\frac{T-1}{T} 1_{\{R(T-1)=1\}}, \frac{1}{T}\right\} \\ &= \frac{T-1}{T} 1_{\{R(T-1)=1\}} + \frac{1}{T} 1_{\{R(T-1)=0\}}. \end{aligned}$$

By induction, we show that—at least for n small enough—

$$V(T-n) = \frac{T-n}{T} 1_{\{R(T-n)=1\}} + v_{T-n} 1_{\{R(T-n)=0\}},$$

where v_t is given recursively by $v_T = 0$ and $v_{t-1} = \frac{1}{T} + \frac{t-1}{t} v_t$ for $t = T-1, T-2, \ldots$ Indeed,

$$\begin{aligned} V(T-n) = &\max\left\{\widetilde{X}(T-n), E(V(T-n+1)|\mathscr{F}_{T-n})\right\} \\ = &\max\Bigg\{\frac{T-n}{T} 1_{\{R(T-n)=1\}}, \\ &\frac{T-n+1}{T} P(R(T-n+1) = 1) + v_{T-n+1} P(R(T-n+1) = 0)\Bigg\} \\ = &\max\left\{\frac{T-n}{T} 1_{\{R(T-n)=1\}}, \frac{T-n+1}{T} \frac{1}{T-n+1} + v_{T-n+1} \frac{T-n}{T-n+1}\right\} \end{aligned}$$

$$
\begin{aligned}
&= \max\left\{\frac{T-n}{T}1_{\{R(T-n)=1\}}, v_{T-n}\right\} \\
&= \frac{T-n}{T}1_{\{R(T-n)=1\}} + v_{T-n}1_{\{R(T-n)=0\}}.
\end{aligned}
$$

However, the last equality only holds as long as $v_{T-n} \leq \frac{T-n}{T}$. Otherwise the second term in the maximum is larger, regardless of the value of $R(T-n)$. Therefore we obtain

$$
V(t) = \begin{cases} v_{t_0}, & t \leq t_0, \\ \frac{t}{T}1_{\{R(t)=1\}} + v_t 1_{\{R(t)=0\}}, & t > t_0, \end{cases}
$$

where t_0 is the largest integer such that $\frac{t_0}{T} < v_{t_0}$. Proposition 1.55 together with the above derivation yields that it is optimal to stop at

$$
\tau := \inf\{t > t_0 : R(t) = 1\} \wedge T
$$

and $P(N(\tau) = 1) = E(\widetilde{X}(\tau)) = V(0) = v_{t_0}$.

The solution to the recursion for v_t is

$$
v_t = \frac{t}{T}\sum_{s=t}^{T-1}\frac{1}{s}
$$

as one can easily verify. Hence t_0 is the largest integer t with $\sum_{s=t}^{T-1}\frac{1}{s} > 1$. For large T we obtain the approximation

$$
\sum_{s=t}^{T-1}\frac{1}{s} = \sum_{s=t}^{T-1} T^{-1}\frac{1}{(s/T)} \approx \int_{t/T}^{1}\frac{1}{s}ds = -\log\frac{t}{T}.
$$

The threshold condition $\sum_{s=t_0}^{T-1}\frac{1}{s} \approx 1$ can then be rephrased as $-\log\frac{t_0}{T} \approx 1$ or

$$
\frac{t_0}{T} \approx \frac{1}{e} \approx 0.368,
$$

which leads to $v_{t_0} \approx -e^{-1}\log e^{-1}$ and hence

$$
v_{t_0} \approx \frac{1}{e} \approx 0.368.
$$

Put differently, it is approximately optimal to let 37% of all candidates pass and then pick the next one that is better than all which have been observed so far. This strategy leads to the globally optimal candidate with approximately 37% probability, which may seem surprisingly high if the total number T of candidates amounts to