STOCHASTIC MODELLING AND APPLIED PROBABILITY

57

Søren Asmussen
Peter W. Glynn

# Stochastic Simulation

## Algorithms and Analysis

Springer

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# Stochastic Modelling and Applied Probability

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*(continued after index)*

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# Preface

Sampling-based computational methods have become a fundamental part of the numerical toolset of practitioners and researchers across an enormous number of different applied domains and academic disciplines. *This book is intended to provide a broad treatment of the basic ideas and algorithms associated with sampling-based methods, often also referred to as* Monte Carlo algorithms *or as* stochastic simulation. The reach of these ideas is illustrated here by discussing a wide range of different applications. Our goal is to provide coverage that reflects the richness of both the applications and the models that have found wide usage.

Of course, the models that are used differ widely from one discipline to another. Some methods apply across the entire simulation spectrum, whereas certain models raise particular computational challenges specific to those model formulations. As a consequence, the first part of the book focuses on general methods, whereas the second half discusses model-specific algorithms. The mathematical level is intended to accommodate the reader, so that for models for which even the model formulation demands some sophistication on the part of the reader (e.g., stochastic differential equations), the mathematical discussion will be at a different level from that presented elsewhere. While we deliver an honest discussion of the basic mathematical issues that arise in both describing and analyzing algorithms, we have chosen not to be too fussy with regard to providing precise conditions and assumptions guaranteeing validity of the stated results. For example, some theorem statements may omit conditions (such as moment hypotheses) that, while necessary mathematically, are not key to understanding the practical domain of applicability of the result. Likewise, in some arguments, we have provided an outline of the key mathematical steps necessary to understand (for example) a rate of convergence issue, without giving all the mathematical details that would serve to provide a complete and rigorous proof.

As a result, we believe that this book can be a useful simulation resource to readers with backgrounds ranging from an exposure to introductory probability to a much more advanced knowledge of the area. Given the wide range of examples and application areas addressed, our expectation is that students, practitioners, and researchers in statistics, probability, operations research, economics, finance, engineering, biology, chemistry, and physics will find the book to be of value. In addition to providing a development of the area pertinent to each reader's specific interests, our hope is that the book also serves to broaden our audience's view of both Monte Carlo and stochastic modeling, in general.

There exists an extensive number of texts on simulation and Monte Carlo methods. Classical general references in the areas covered by this book are (in chronological order) Hammersley & Handscombe [173], Rubinstein [313], Ripley [300], and Fishman [118]. A number of further ones can be found in the list of references; many of them contain much practically oriented discussion not at all covered by this book. There are further a number of books dealing with special subareas, for example Gilks et al. [129] on Markov chain Monte Carlo methods, Newman & Barkema [276] on applications to statistical physics, Glasserman [133] on applications to mathematical finance, and Rubinstein & Kroese [318] on the cross-entropy method.

In addition to standard journals in statistics and applied probability, the reader interested in pursuing the literature should be aware of journals like *ACM TOMACS* (*ACM Transactions of Modeling and Computer Simulation*), *Management Science*, and the *IEEE* journals. Of course, today systematic scans of journals are to a large extent replaced by searches on the web. At the end of the book after the References section, we give some selected web links, being fully aware that such a list is likely to be outdated soon. These links also point to some important recurrent conferences on simulation, see in particular [$\mathtt{w}^3$.14], [$\mathtt{w}^3$.16], [$\mathtt{w}^3$.17], [$\mathtt{w}^3$.20].

The book is designed as a potential teaching and learning vehicle for use in a wide variety of courses. Our expectation is that the appropriate selection of material will be highly discipline-dependent, typically covering a large portion of the material in Part A on general methods and using those special topics chapters in Part B that reflect the models most widely used within that discipline. In teaching this material, we view some assignment of computer exercises as being essential to gaining an understanding and intuition for the material. In teaching graduate students from this book, one of us (SA) assigns a computer lab of three hours per week to complement lectures of two hours per week. Exercises labeled (A) are designed for such a computer lab (although whether three hours is sufficient will depend on the students, and certainly some home preparation is needed). We have also deliberately chosen to not focus the book on a specific simulation language or software environment. Given the broad range of models covered, no single programming environment would provide a good universal fit. We prefer to let the user or teacher make the software choice herself. Finally, as a matter of teaching philosophy, we do not believe that programming should take a central role in a course taught from this book. Rather, the focus should be on understanding the intuition underlying the algorithms described here, as well as their strengths and weaknesses. In fact, to avoid a focus on the programming per se, we often hand out pieces of code for parts that are tedious to program but do not involve advanced ideas. Exercises marked (TP) are theoretical problems, highly varying in difficulty.

Since the first slow start of the writing of this book in 1999, we have received a large number of useful comments, suggestions, and corrections on earlier version of the manuscript. Thanks go first of all to the large number of students who have endured coping with these early versions. It would go too far to mention all the colleagues who have helped in one way or another. However, for a detailed reading of larger parts it is a pleasure to thank Hansjörg Albrecher, Morten Fenger-Grøn, Pierre L'Ecuyer, Thomas Mikosch, Leonardo Rojas-Nandayapa, and Jan Rosiński. At the technical level, Lars Madsen helped with many problems that were beyond our LaTeX ability.

A list of typos will be kept at [w$^3$.1], and we are greatful to be informed of misprints as well as of more serious mistakes and omissions.

Aarhus and Stanford
February 2007

Søren Asmussen
Peter W. Glynn

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# Notation

### *Internal Reference System*

The chapter number is specified only if it is not the current one. As examples, Proposition 1.3, formula (5.7) or Section 5 of Chapter IV are referred to as IV.1.3, IV.(5.7) and IV.5, respectively, in all chapters other than IV where we write Proposition 1.3, formula (5.7) (or just (5.7)) and Section 5.

### *Special Typeface*

| | |
|---|---|
| $\mathrm{d}$ | differential like in $\mathrm{d}x$, $\mathrm{d}t$, $F(\mathrm{d}x)$; to be distinguished from a variable or constant $d$, a function $d(x)$ etc. |
| $\mathrm{e}$ | the base $2.71\ldots$ of the natural logarithm; to be distinguished from $e$ which can be a variable or a different constant. |
| $\mathrm{i}$ | the imaginary unit $\sqrt{-1}$; to be distinguished from a variable $i$ (typically an index). |
| $\mathbb{1}$ | the indicator function, for example $\mathbb{1}_A$, $\mathbb{1}_{x\in A}$, $\mathbb{1}\{x \in A\}$, $\mathbb{1}\{X(t) > 0 \text{ for some } t \in [0,1]\}$. |
| $\mathrm{O}$, $\mathrm{o}$ | the Landau symbols. That is, $f(x) \,.\!= \mathrm{O}\big(g(x)\big)$ means that $f(x)/g(x)$ stays bounded in some limit, say $x \to \infty$ or $x \to 0$, whereas $f(x) = \mathrm{o}\big(g(x)\big)$ means $f(x)/g(x) \to 0$. |
| $\mathrm{\pi}$ | $3.1416\ldots$; to be distinguished from $\pi$ which is often used for a stationary distribution or other. |

| | |
|---|---|
| $\mathscr{N}(\mu, \sigma^2)$ | the normal distribution with mean $\mu$ and variance $\sigma^2$. |

Probability, expectation, variance, covariance are denoted $\mathbb{P}$, $\mathbb{E}$, $\mathbb{V}ar$, $\mathbb{C}ov$. The standard sets are $\mathbb{R}$ (the real line $(-\infty, \infty)$), the complex numbers $\mathbb{C}$, the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$, the integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$. Matrices and vectors are most often denoted by bold typeface, $\boldsymbol{C}$, $\boldsymbol{\Sigma}$, $\boldsymbol{x}$, $\boldsymbol{\alpha}$ etc., though exceptions occur. The transpose of $\boldsymbol{A}$ is denoted $\boldsymbol{A}^{\mathsf{T}}$.

## *Miscellaneous Mathematical Notation*

| | |
|---|---|
| $\stackrel{\text{def}}{=}$ | a defining equality. |
| $\stackrel{\text{a.s.}}{\to}$ | a.s. convergence |
| $\stackrel{\mathbb{P}}{\to}$ | convergence in probability |
| $\stackrel{\mathscr{D}}{\to}$ | convergence in distribution |
| $\stackrel{\mathscr{D}}{=}$ | equality in distribution |
| $\longleftarrow$ | an assignment in an algorithm (not used throughout) |
| $\lvert \cdot \rvert$ | in addition to absolute value, also used for the number of elements (cardinality) $\lvert S \rvert$ of a set $S$, or its Lebesgue measure $\lvert S \rvert$. |
| $\mathbb{E}[X; A]$ | $\mathbb{E}[X \mathbb{1}_A]$. |
| $\sim$ | usually, $a(x) \sim b(x)$ means $a(x)/b(x) \to 1$ in some limit like $x \to 0$ or $x \to \infty$, but occassionally, other posssibilities occur. E.g. $X \sim \mathscr{N}(\mu, \sigma^2)$ specifies $X$ to have a $\mathscr{N}(\mu, \sigma^2)$ distribution. |
| $\approx$ | a different type of asymptotics, often just at the heuristical level. |
| $\stackrel{\mathscr{D}}{\approx}$ | approximate equality in distribution. |
| $\propto$ | proportional to. |
| $\widehat{F}[\cdot]$ | the m.g.f. of a distribution $F$. Thus $\widehat{F}[\mathrm{i}s]$ is the characteristic function at $s$. Sometimes $\widehat{F}[\cdot]$ is also used for the probability generating function of a discrete r.v. |

The letter $U$ is usually reserved for a uniform$(0, 1)$ r.v., and the letter $z$ for a quantity to be estimated by simulation, $Z$ for a r.v. with $\mathbb{E}Z = z$. As is standard, $\Phi$ is used for the c.d.f. of $\mathscr{N}(0, 1)$ and $\varphi(x) \stackrel{\text{def}}{=} \mathrm{e}^{-x^2/2}/\sqrt{2\pi}$ for the density.. $z_\alpha$ often denotes the $\alpha$-quantile of $\mathscr{N}(0, 1)$. A standard Brownian motion is denoted $B$ and one with possibly drift $\mu \neq 0$ and/or variance $\sigma^2$ by W. Exceptions to all of this occur occasionally.