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Stochastic Linear-Quadratic Optimal Control Theory: Differential Games and Mean-Field Problems

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To Our Parents

Yuqi Sun and Xiuying Ma

Wenyao Yong and Xiangxia Chen

Preface

Linear-quadratic optimal control theory (LQ theory, for short) has a long history, and most people feel that the LQ theory is quite mature. Three well-known relevant issues are involved: existence of optimal controls, solvability of the optimality system (which is a two-point boundary value problem), and solvability of the associated Riccati equation. A rough impression is that these three issues are somehow equivalent.

In the past few years, we, together with our collaborators, have been re-investigating the LQ theory for stochastic systems with deterministic coefficients. A number of interesting delicate issues have been identified, including:

- For finite-horizon LQ problems, open-loop optimal controls and closed-loop optimal strategies should be distinguished because the existence of the latter implies the existence of the former, but not vice versa. Whereas, for infinite-horizon LQ problems, under proper conditions, the open-loop and closed-loop solvability are equivalent.
- For finite-horizon two-person (not necessarily zero-sum) differential games, the open-loop and closed-loop Nash equilibria are two different concepts. The existence of one of them does not imply the existence of the other, which is different from LQ optimal control problems.
- The closed-loop representation of an open-loop Nash equilibrium is not necessarily the outcome of a closed-loop Nash equilibrium.

Our investigations also revealed some previously unknown facts concerning two-person differential games. A partial list is:

- For two-person (not necessarily zero-sum) differential games in finite horizons, the existence of an open-loop Nash equilibrium is equivalent to the solvability of a system of coupled FBSDEs, together with the convexities of the cost functionals; the existence of a closed-loop Nash equilibrium is equivalent to the solvability of a Lyapunov-Riccati type equation.

- For two-person zero-sum differential games, both in finite and infinite horizons, if closed-loop saddle points exist, and an open-loop saddle point exists and admits a closed-loop representation, then the representation must be the outcome of some closed-loop saddle point. Such a result also holds for LQ optimal control problems.
- For two-person zero-sum differential games over an infinite horizon, the existence of an open-loop and a closed-loop Nash equilibrium are equivalent.
- Some of the results concerning LQ optimal control problems can further be extended to the case when expectations of the state and the control are involved. This kind of LQ problems is referred to as the mean-field problem.

The purpose of this book is to systematically present the above-mentioned results concerning LQ differential games and mean-field LQ optimal control problems. We assume that readers are familiar with basic stochastic analysis and stochastic control theory.

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Shenzhen, China Jingrui Sun
Orlando, USA Jiongmin Yong
March 2020

Contents

Frequently Used Notation

I. Notation for Euclidean Spaces and Matrices

1. $\mathbb{R}^{n\times m}$: the space of all $n\times m$ real matrices.
2. $\mathbb{R}^n=\mathbb{R}^{n\times 1}$; $\mathbb{R}=\mathbb{R}^1$; $\overline{\mathbb{R}}=[-\infty,\infty]$.
3. $\mathbb{S}^n$: the space of all symmetric $n\times n$ real matrices.
4. $\mathbb{S}^n_+$: the subset of $\mathbb{S}^n$ consisting of positive definite matrices.
5. $\bar{\mathbb{S}}^n_+$: the subset of $\mathbb{S}^n$ consisting of positive semi-definite matrices.
6. I_n: the identity matrix of size n, which is also denoted simply by I if no confusion occurs.
7. $M^\top$: the transpose of a matrix M.
8. $M^\dagger$: the Moore-Penrose pseudoinverse of a matrix M.
9. $\mathrm{tr}(M)$: the sum of diagonal elements of a square matrix M, called the trace of M.
10. $\langle\cdot,\cdot\rangle$: the inner product on a Hilbert space. In particular, the usual inner product on $\mathbb{R}^{n\times m}$ is given by $\langle M,N\rangle\mapsto\mathrm{tr}\big(M^\top N\big)$.
11. $|M|\triangleq\sqrt{\mathrm{tr}(M^\top M)}$: the Frobenius norm of a matrix M.
12. $\mathscr{R}(M)$: the range of a matrix or an operator M.
13. $\mathscr{N}(M)$: the kernel of a matrix or an operator M.
14. $A\geqslant B$: $A-B$ is a positive semi-definite symmetric matrix.
15. $\mathcal{Q}(P)\triangleq PA+A^\top P+C^\top PC+Q$.
16. $\mathcal{S}(P)\triangleq B^\top P+D^\top PC+S$.
17. $\mathcal{R}(P)\triangleq R+D^\top PD$.
18. $\widehat{A}\triangleq A+\bar{A}$, $\widehat{B}\triangleq B+\bar{B}$, $\widehat{C}\triangleq C+\bar{C}$, $\widehat{D}\triangleq D+\bar{D}$.
19. $\widehat{Q}\triangleq Q+\bar{Q}$, $\widehat{S}\triangleq S+\bar{S}$, $\widehat{R}\triangleq R+\bar{R}$, $\widehat{G}\triangleq G+\bar{G}$.
20. $\widehat{\mathcal{Q}}(P,\Pi)\triangleq\Pi\widehat{A}+\widehat{A}^\top\Pi+\widehat{C}^\top P\widehat{C}+\widehat{Q}$.
21. $\widehat{\mathcal{S}}(P,\Pi)\triangleq\widehat{B}^\top\Pi+\widehat{D}^\top P\widehat{C}+\widehat{S}$.
22. $\widehat{\mathcal{R}}(P)\triangleq\widehat{R}+\widehat{D}^\top P\widehat{D}$.

II. Sets and Spaces of Functions and Processes

Let $\mathbb{H}$ be a Euclidian space (which could be $\mathbb{R}^n$, $\mathbb{R}^{n\times m}$, etc.).

1. $C([t,T];\mathbb{H})$: the space of $\mathbb{H}$-valued, continuous functions on $[t,T]$.
2. $L^p(t,T;\mathbb{H})$: the space of $\mathbb{H}$-valued functions that are pth $(1\leqslant p<\infty)$ power Lebesgue integrable on $[t,T]$.
3. $L^\infty(t,T;\mathbb{H})$: the space of $\mathbb{H}$-valued, Lebesgue measurable functions that are essentially bounded on $[t,T]$.
4. $L^2_{\mathcal{F}_t}(\Omega;\mathbb{H})$: the space of $\mathcal{F}_t$-measurable, $\mathbb{H}$-valued random variables ξ such that $\mathbb{E}|\xi|^2<\infty$.
5. $L^2_{\mathbb{F}}(\Omega;L^1(t,T;\mathbb{H}))$: the space of $\mathbb{F}$-progressively measurable, $\mathbb{H}$-valued processes $\varphi:[t,T]\times\Omega\to\mathbb{H}$ such that $\mathbb{E}\Big[\int_t^T|\varphi(s)|ds\Big]^2<\infty$.
6. $L^2_{\mathbb{F}}(t,T;\mathbb{H})$: the space of $\mathbb{F}$-progressively measurable, $\mathbb{H}$-valued processes $\varphi:[t,T]\times\Omega\to\mathbb{H}$ such that $\mathbb{E}\int_t^T|\varphi(s)|^2ds<\infty$.
7. $L^2_{\mathbb{F}}(\mathbb{H})$: the space of $\mathbb{F}$-progressively measurable, $\mathbb{H}$-valued processes $\varphi:[0,\infty)\times\Omega\to\mathbb{H}$ such that $\mathbb{E}\int_0^\infty|\varphi(t)|^2dt<\infty$.
8. $L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{H}))$: the space of $\mathbb{F}$-adapted, continuous, $\mathbb{H}$-valued processes $\varphi:[t,T]\times\Omega\to\mathbb{H}$ such that $\mathbb{E}\Big[\sup_{s\in[t,T]}|\varphi(s)|^2\Big]<\infty$.
9. $\mathcal{X}_t=L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$.
10. $\mathcal{X}[t,T]=L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{R}^n))$.
11. $\mathcal{U}[t,T]=L^2_{\mathbb{F}}(t,T;\mathbb{R}^m)$.
12. $\mathcal{X}_{loc}[0,\infty)=\bigcap_{T>0}\mathcal{X}[0,T]$.
13. $\mathcal{X}[0,\infty)$: the subspace of $\mathcal{X}_{loc}[0,\infty)$ consisting of processes φ which are square-integrable: $\mathbb{E}\int_0^\infty|\varphi(t)|^2dt<\infty$.

Chapter 1
Some Elements of Linear-Quadratic Optimal Controls

Abstract This chapter is a brief review on the stochastic linear-quadratic optimal control. Some useful concepts and results, which will be needed throughout this book, are presented in the context of finite and infinite horizon problems. These materials are mainly for beginners and may also serve as a quick reference for knowledgeable readers.

Keywords Linear-quadratic · Optimal control · Finite horizon · Infinite horizon · Riccati equation · Open-loop · Closed-loop

In this chapter, we briefly review the stochastic linear-quadratic (LQ, for short) optimal control problem and present some useful concepts and results in the context of finite and infinite horizon problems. Most of the results recalled here are quoted from the book [48] by Sun and Yong, where rigorous proofs can be found. In the sequel, $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a complete probability space on which a standard one-dimensional Brownian motion $W = \{W(t); 0 \leqslant t < \infty\}$ is defined, and $\mathbb{F}$ denotes the usual augmentation of the natural filtration $\{\mathcal{F}_t\}_{t \geqslant 0}$ generated by W. For a random variable ξ, we write $\xi \in \mathcal{F}_t$ if ξ is $\mathcal{F}_t$-measurable, and for a stochastic process φ, we write $\varphi \in \mathbb{F}$ if it is $\mathbb{F}$-progressively measurable.

1.1 LQ Optimal Control Problems in Finite Horizons

Consider the following controlled linear stochastic differential equation (SDE, for short) on a finite horizon $[t, T]$:

$$\begin{cases} dX(s) = [A(s)X(s) + B(s)u(s) + b(s)]ds \\ \qquad\qquad + [C(s)X(s) + D(s)u(s) + \sigma(s)]dW(s), \\ X(t) = x, \end{cases} \tag{1.1.1}$$

where $A, C : [0, T] \to \mathbb{R}^{n\times n}$, $B, D : [0, T] \to \mathbb{R}^{n\times m}$, called the *coefficients* of the *state equation* (1.1.1), and $b, \sigma : [0, T] \times \Omega \to \mathbb{R}^n$, called the *nonhomogeneous terms*, satisfy the following assumption.

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(H1) The coefficients and the nonhomogeneous terms of (1.1.1) satisfy

$$\begin{cases} A \in L^1(0,T;\mathbb{R}^{n\times n}), & B \in L^2(0,T;\mathbb{R}^{n\times m}), & b \in L^2_{\mathbb{F}}(\Omega;L^1(0,T;\mathbb{R}^n)), \\ C \in L^2(0,T;\mathbb{R}^{n\times n}), & D \in L^\infty(0,T;\mathbb{R}^{n\times m}), & \sigma \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n). \end{cases}$$

In the above assumption we have adopted the following notation: For a subset $\mathbb{H}$ of some Euclidean space (which could be $\mathbb{R}^n$, $\mathbb{R}^{n\times m}$, etc.),

$$\begin{aligned} L^p(t,T;\mathbb{H}) &= \Big\{\varphi:[t,T]\to\mathbb{H} \;\Big|\; \int_t^T |\varphi(s)|^p ds < \infty\Big\} \quad (1 \leqslant p < \infty), \\ L^\infty(t,T;\mathbb{H}) &= \Big\{\varphi:[t,T]\to\mathbb{H} \;\Big|\; \varphi \text{ is essentially bounded}\Big\}, \\ L^2_{\mathbb{F}}(t,T;\mathbb{H}) &= \Big\{\varphi:[t,T]\times\Omega\to\mathbb{H} \;\Big|\; \varphi\in\mathbb{F}, \mathbb{E}\int_t^T |\varphi(s)|^2 ds < \infty\Big\}, \\ L^2_{\mathbb{F}}(\Omega;L^1(t,T;\mathbb{H})) &= \Big\{\varphi:[t,T]\times\Omega\to\mathbb{H} \;\Big|\; \varphi\in\mathbb{F}, \mathbb{E}\Big[\int_t^T |\varphi(s)| ds\Big]^2 < \infty\Big\}. \end{aligned}$$

The process $u = \{u(s); t \leqslant s \leqslant T\}$ in (1.1.1) belongs to the space

$$\mathcal{U}[t,T] \equiv L^2_{\mathbb{F}}(t,T;\mathbb{R}^m)$$

and is called a *control*.

The *cost functional* associated with the state equation (1.1.1) is of the quadratic form

$$\begin{aligned} J(t,x;u) = \mathbb{E}\Big\{&\langle GX(T),X(T)\rangle + 2\langle g, X(T)\rangle \\ &+ \int_t^T \Big[\Big\langle \begin{pmatrix} Q(s) & S(s)^\top \\ S(s) & R(s)\end{pmatrix}\begin{pmatrix} X(s) \\ u(s)\end{pmatrix}, \begin{pmatrix} X(s) \\ u(s)\end{pmatrix}\Big\rangle \\ &+ 2\Big\langle \begin{pmatrix} q(s) \\ \rho(s)\end{pmatrix}, \begin{pmatrix} X(s) \\ u(s)\end{pmatrix}\Big\rangle\Big] ds\Big\}. \end{aligned} \tag{1.1.2}$$

In the above, $\langle\cdot\,,\cdot\rangle$ stands for the Frobenius inner product of two matrices that have the same size. That is, for $M, N \in \mathbb{R}^{n\times m}$, $\langle M, N\rangle$ is equal to the trace of $M^\top N$. The superscript $\top$ denotes the transpose of matrices. In the sequel, the identity matrix of size n will be denoted by I_n, and the Frobenius norm of a matrix M will be denoted by $|M|$. Let $\mathbb{S}^n$ (respectively, $\mathbb{S}^n_+$) be the space of all symmetric $n\times n$ real matrices (respectively, positive definite matrices), and let

$$L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n) = \big\{\xi:\Omega\to\mathbb{R}^n \mid \xi \text{ is } \mathcal{F}_t\text{-measurable with } \mathbb{E}|\xi|^2 < \infty\big\}.$$

The *weighting matrices* in the cost functional are assumed to satisfy the following condition.