Part I

FINANCIAL MARKETS AND POPULAR MODELS

1 Financial Markets – Data, Basics and Derivatives

1.1 INTRODUCTION AND OBJECTIVES

The first chapter is to introduce the models that appear in subsequent chapters and, in so doing, to highlight the necessity of applying advanced numerical techniques. Since we wish to apply mathematical models to financial problems, we first have to analyse the markets under consideration. We have to check the available data upon which we build our models. Then, we have to investigate which models are appropriate and, finally, we need to decide on numerical methods to solve the modelling problem.

We motivate using market data; we highlight the nature of risk and the problems which arise with inappropriate modelling. The final conclusion is that the observed market structures need sophisticated models, numerically challenging implementation and deeply involved special purpose algorithms. Furthermore, we provide answers and suggestions to the following questions:

- What kind of objects do we have to model?
- What kind of distributions are necessary? Do we need anything other than the Gaussian distribution?
- What kind of patterns do we observe and which model is capable of reproducing such patterns?
- How complex should a model be?
- Which mathematical methods do we need? PDE? SDE? Numerical Mathematics?

We do not rate the models, but we do give advice on the numerical methods which can be applied to implement the different models and on what kind of market observation is covered by a certain model. We work out several methods which can be applied. The reader can try the different solutions and – very important – check the implementation, the stability and the robustness. Furthermore, the code provided can be modified to fit the special modelling issues.

Since financial models have to be implemented as computer programs, or they have to be integrated into a pricing library, numerical methods are required. The most fundamental risk of a model is, of course, its inapplicability in a certain setting. To this end we have to analyse which risk factors can be modelled using a certain class of models and we have to be aware of the risk factors that have not been taken into account. But once we have decided to apply a particular model, and we think that we are applying it appropriately, we face the following challenges:

- Appropriate numerical techniques.
- Approximations used should be robust, efficient and accurate.
- Black box solutions should be avoided.
- The implementation should be stable and reliable.

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1.2 FINANCIAL TIME-SERIES, STATISTICAL PROPERTIES OF MARKET DATA AND INVARIANTS

To use a mathematical model for gaining insights and applying it to financial market data we need to choose some quantities or risk factors which we model. To this end we consider the notion of a *market invariant*. Fix a starting point t_{start} in time and an estimation interval τ. The interval τ could be one day or one month, for instance. Suppose from a market data provider we can get the data for an index $X(t)$, $t \in \mathcal{T}$, with

$$\mathcal{T} := \{t_{\text{start}}, t_{\text{start}} + \tau, \ldots, t_{\text{start}} + n \cdot \tau\}, \quad n \in \mathbb{N}.$$

We regard $X(t)$ as a random variable. A random variable X is called a market invariant for t_{start} and estimation interval τ if the realizations

- are *independent*
- are *identically distributed*.

A simple but effective method to test if a random variable qualifies as an invariant is the following:

- Take a time series X_s, $s = t, t + \tau, \ldots, t + n \cdots \tau = T$ of the possible invariant.
- Split the time series into two parts

$$\begin{aligned} X_t^1 &= x_t^1, \quad t \in \{t_{\text{start}}, \ldots, [T - t_{\text{start}}/2\tau] \cdot \tau\} \\ X_t^2 &= x_t^2, \quad t \in \{([T - t_{\text{start}}/2\tau] + 1) \cdot \tau, \ldots, T\}. \end{aligned}$$

- Plot histograms corresponding to X^1 and X^2.
- Plot lagged time series $\tilde{X}_t := X_{t-\tau}$ against X_t.

Let us illustrate this test on time series for index and swap data. Before we actually start let us illustrate the dependence structure corresponding to independent, positively and negatively dependent random variables. To this end we take as an example the normal distribution with zero mean and a given covariance matrix, Σ. For our examples we choose three different covariance matrices, namely,

$$\Sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Sigma_1 = \begin{pmatrix} 1 & 0.99 \\ 0.99 & 1 \end{pmatrix}, \quad \Sigma_2 = \begin{pmatrix} 1 & -0.99 \\ -0.99 & 1 \end{pmatrix}.$$

The dependence structure is displayed in Figure 1.1.

We call a market invariant X *time homogeneous* if the distribution of X does not depend on the chosen time point t_{start}. In the sequel we consider Equity, Index, Interest Rate and Option markets. First, we consider index time series for the *S&P 500*, the *Nikkei*, the *FTSE* and the *DAX*. We argue that the prices of the indices do not obey the properties necessary to be an invariant.

The first observation regarding the data is that if we plot the lagged time series directly we get Figure 1.2. This clearly shows that the plain data are not independent and therefore not an invariant.

Furthermore, when we plot histograms with respect to the observed data we cannot find a suitable distributional description. Figure 1.3 shows the corresponding histograms.

Now, we consider the logarithmic returns computed from the time series. We see a very different picture. Figures 1.4 and 1.5 suggest that these quantities are invariants.

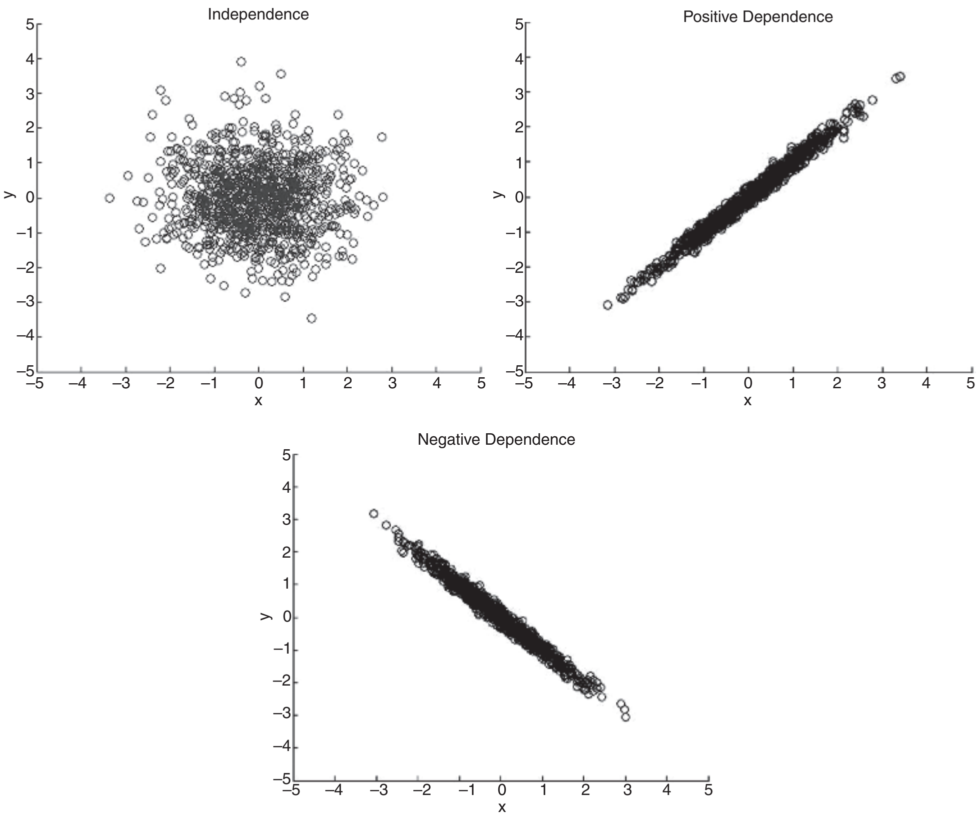

Figure 1.1 Time series generated from a normal distribution with covariance given by Σ_0 (top left), Σ_1 (top right) and Σ_2 (bottom) reflecting independence, positive and negative dependence

Thus, without further discussion we take as a suitable choice for a market invariant in the equity market the *logarithmic returns*, given by

$$H(t, \tau) := \log\left(\frac{S(t+\tau)}{S(t)}\right). \tag{1.1}$$

In fact, let $g : \mathbb{R} \to \mathbb{R}$ be a function, then $g(H)$ is a market invariant.

Taking realized prices of an index it is difficult to assign probabilistic concepts. It is not clear how to obtain relevant statistical information using such prices. On the contrary, the logarithmic returns introduced in Equation (1.1) show that the observed time series are independent and some parametric probability distribution can be assigned.

Other market invariants can also be derived. For a general and formal treatment see Meucci, A. (2007). We consider the case of the interest rate market. The zero coupon bonds, $DF(t, T)$ and the ratio $\frac{DF(t,T)}{DF(t-\tau,T)}$ might be considered as invariants. But since they tend to 1, respectively its redemption at expiry, zero coupon bonds are not time-homogeneous and therefore not market invariants. For further illustration let us take non-overlapping total returns, R^v, with

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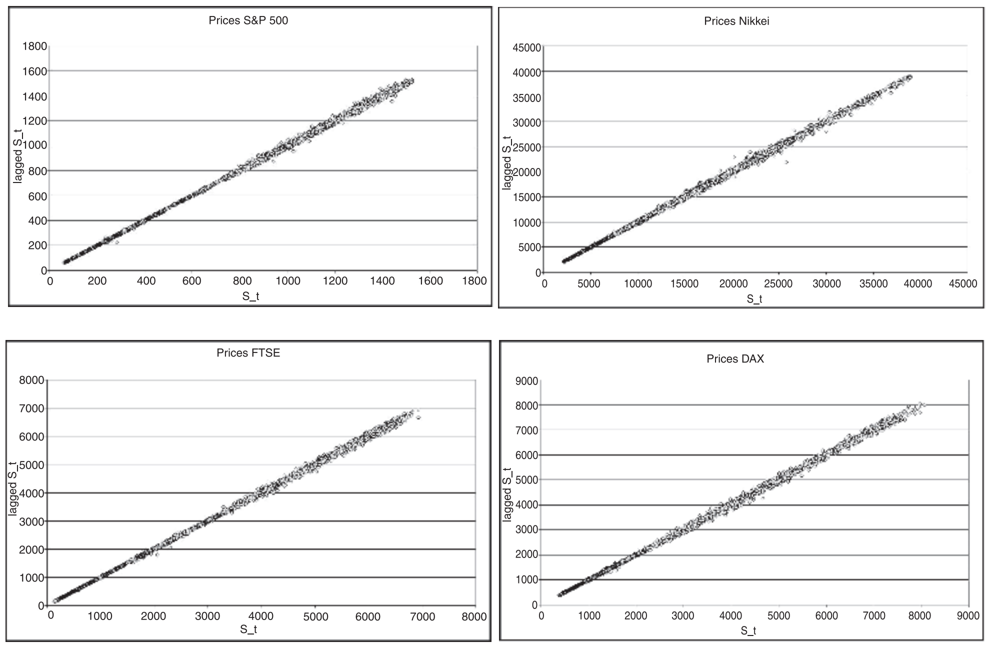

Figure 1.2 Lagged Time Series data of daily index closing prices for S&P 500 (top left), Nikkei (top right), FTSE (bottom left) and DAX (bottom right) calculated from daily index closing prices

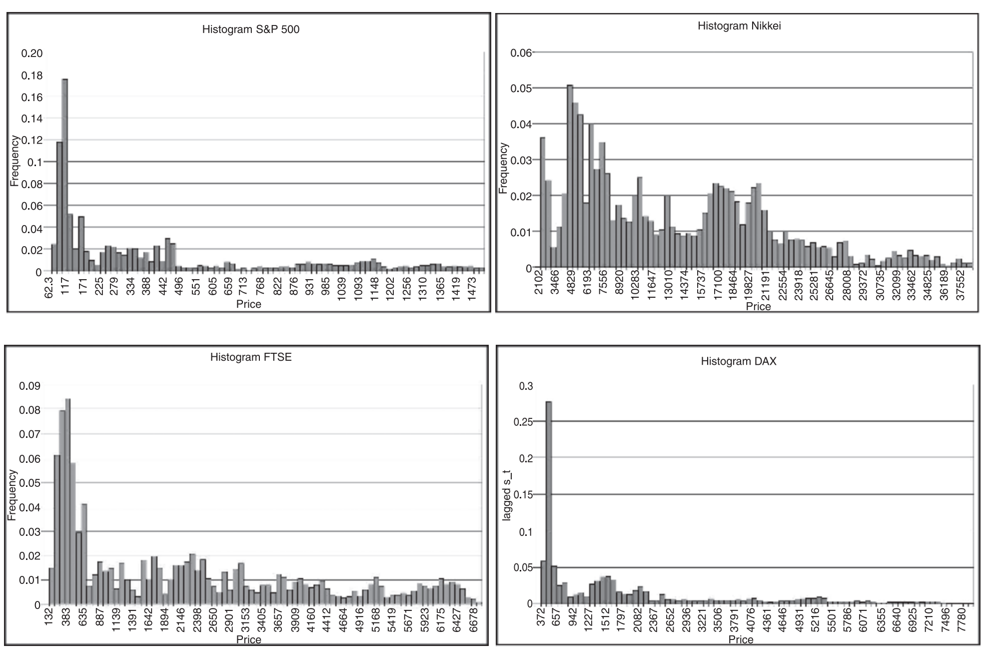

Figure 1.3 Histograms of daily index closing prices for S&P 500 (top left), Nikkei (top right), FTSE (bottom left) and DAX (bottom right)

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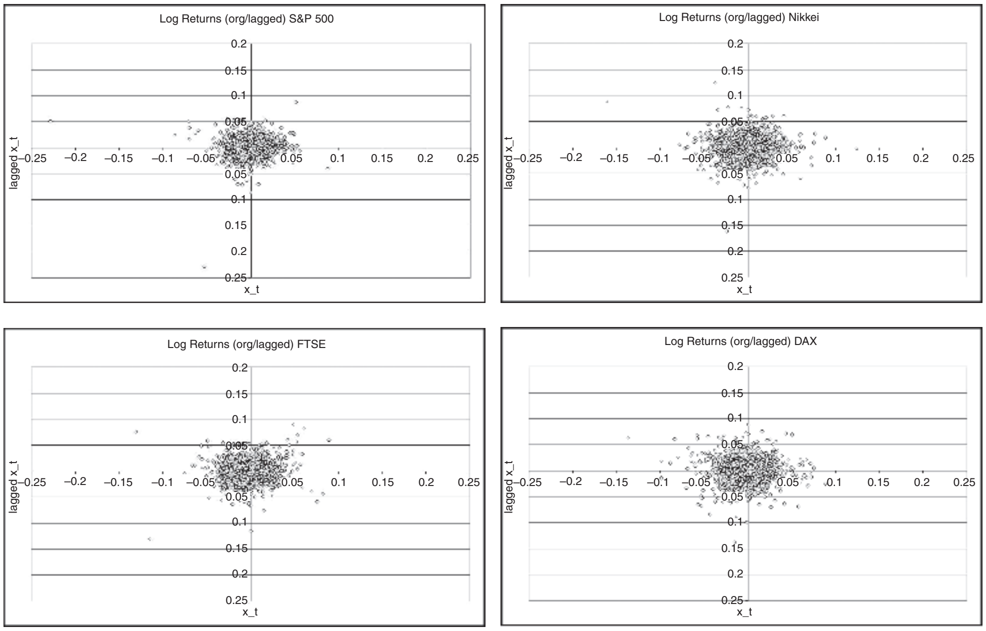

Figure 1.4 Lagged time series for the logarithmic returns for S&P 500 (top left), Nikkei (top right), FTSE (bottom left) and DAX (bottom right) calculated from time series of the daily closing prices

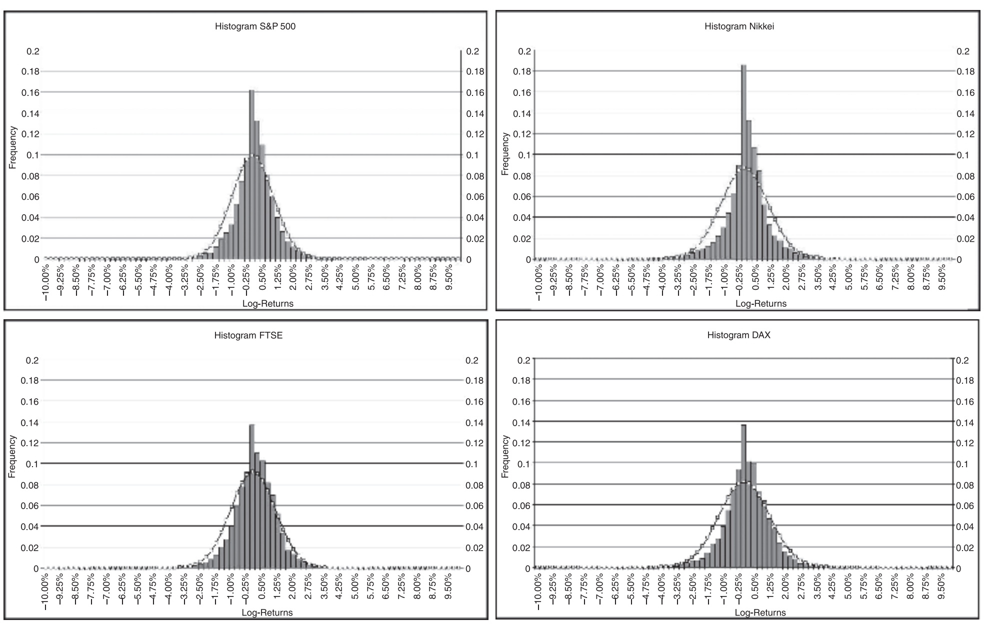

Figure 1.5 Histogram for logarithmic returns for S&P 500 (top left), Nikkei (top right), FTSE (bottom left) and DAX (bottom right) calculated from time series for daily closing prices. Furthermore, the figure shows a moment matched normal distribution

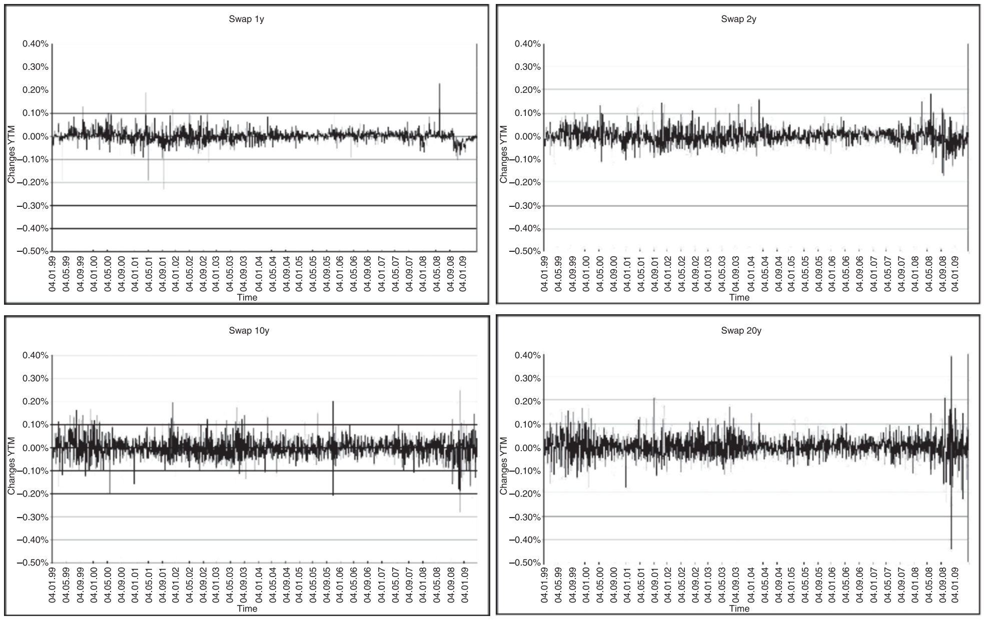

Figure 1.6 Time Series for Changes in Yield to Maturity for 1Y swap rate (top left), 2Y swap rate (top right), 10Y swap rate (bottom left) and 20y swap rate (bottom right)

maturity v. Thus,

$$R^v(t) = \frac{DF(t, t+v)}{DF(t-\tau, t-\tau+v)},$$

and therefore $g(R^v(t))$ are market invariants since they are time homogeneous, independent and identically distributed. A convenient invariant is the change in *yield to maturity ($Y_{2\mathrm{m}}$)* and the changes in $Y_{2\mathrm{m}}$ denoted by Y:

$$\begin{aligned} Y_{2\mathrm{m}}(t, v) &:= -\frac{1}{v}\log\left(DF(t, t+v)\right) \\ Y(t, v, \tau) &:= Y_{2\mathrm{m}}(t, v) - Y_{2\mathrm{m}}(t-\tau, v). \end{aligned}$$

As in the equity market we observe that the time series generated by the changes of the yield to maturity shows the desired properties. The time series of the changes in the yield to maturity derived from different quoted swap rates are plotted in Figure 1.6.

The corresponding historical distributions are plotted in Figure 1.7 and we again see that the assignment of some parametric probability distribution can be achieved.

Another method is to measure the logarithmic returns generated by investing a certain amount of currency using the current quoted rate for a pre-specified period.

Finally, let us consider the option market. For example, for a plain Vanilla Call option, the time value depends on the price of the underlying, on the yield and on the volatility. Therefore, we write the price of a European Call option, C, as

$$C(T-t, K, S(t), r(t), \sigma(t, S(t), K^{\mathrm{ATM}}(t)).$$

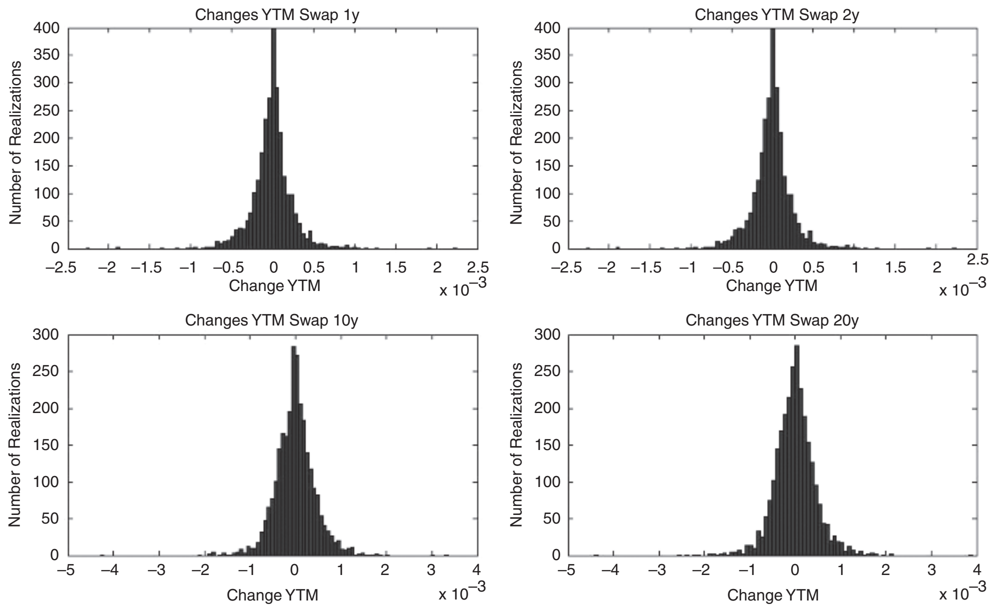

Figure 1.7 Histograms for Changes in Yield to Maturity for 1Y swap rate (top left), 2Y swap rate (top right), 10Y swap rate (bottom left) and 20Y swap rate (bottom right)

For reasons of liquidity we only consider ATM forward volatility. This means that the strike price is equal to the forward $S(T)$. Since we have already identified the invariants for $S(t)$ and $r(t)$ we are left with the problem of determining the invariants for σ. For time homogeneity reasons we consider a fixed time period v and the corresponding maturity $K(t)$

$$\sigma(t, K(t), t + v).$$

The option price fluctuates if the underlying does, but this is not the case for volatility since

$$\sigma \approx \sqrt{\frac{2\pi}{v}} \frac{C(T + v)}{S(T)}.$$

If we take differences in rolling ATM forward volatility:

$$\sigma(t, K^{\text{ATM}}(t), t + v) - \sigma(t - \tau, K^{\text{ATM}}(t - \tau), t - \tau + v).$$

This choice fulfils the constraints on invariants.

1.2.1 Real World Distribution

Suppose we have identified a market invariant such as the logarithmic returns of an index. We are now interested in the historical distribution of the invariant since this can be used as an input to some financial model. This could, for instance, be an asset allocation algorithm.

Figure 1.8 illustrates the difference of the real world distribution of the German DAX index from realized data. Furthermore, we determined the first and the second moment, which are the expectation and the standard deviation. For comparison to the real world distribution we used a moment matched Gaussian distribution. We see that this distribution is not capable of

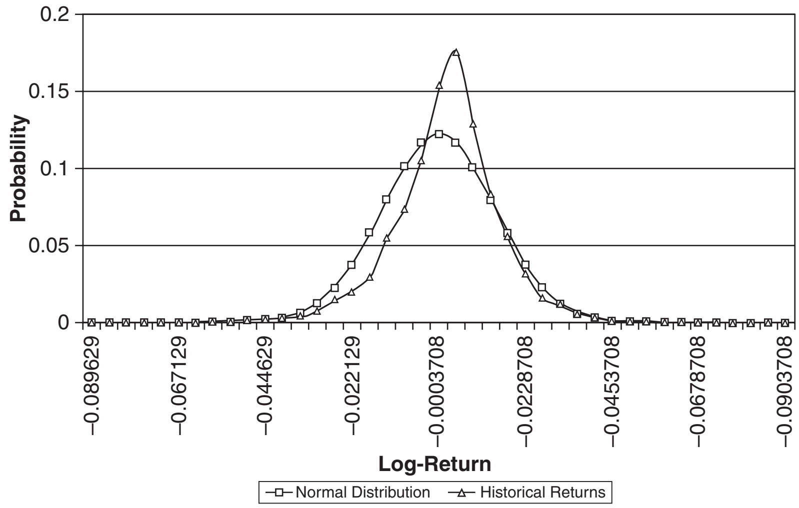

Figure 1.8 DAX index distribution of logarithmic returns and normal density with matched mean and variance

matching the shape of the historical distribution. In what follows we show that calculating risk figures using the Gaussian distribution or using it for asset allocation purposes causes some severe problems.

Applying the Gaussian distribution does not lead to a reasonable description of the risks involved in financial markets. Methods based on calculating risk rely on the computation of *quantiles* of a cumulative distribution. The quantile function is the functional inverse of the cumulative distribution. If $\mathbb{P}$ is the cumulative distribution taking values in [0, 1], we fix a number $\alpha \in [0, 1]$ and determine the value of x for which $\mathbb{P}(x) = \alpha$. Then, x is referred to as the α-quantile. Methods based on the computation of quantiles include Value-at-Risk (VaR), Conditional-Value-at-Risk (CVaR), (1.2), or time series analysis for asset allocation.

We base our consideration on Shaw, W.T. (2011). We define *CVaR* which is also known as *expected shortfall*. This risk figure can be related to VaR by

$$\alpha - \text{CVaR} := \alpha - \frac{\int_{-\infty}^{\alpha} F(x)dx}{F(\alpha)} \geq \alpha - \text{VaR}. \tag{1.2}$$

Let us briefly discuss the inadequacy of using the Gaussian distribution for modelling real world distributions and applying it to identify risk. The real world or statistical distribution is the distribution obtained by time series analysis. Recently, the credit crisis of 2008 showed that the Gaussian assumption for modelling gave rise to many errors.

Often statements of the type “We have seen a sequence of 25 standard deviation events” or “several 10 standard deviation events occurred several days in a row” could be heard in interviews or appeared in the financial press. This illustrates the inadequacy of the Gaussian hypothesis in financial modelling. Statistically, for a 25 standard deviation event to happen it would need longer than the lifetime of the universe. But reality shows that extreme events occur much more often than is suggested by models based on the Gaussian assumption. This

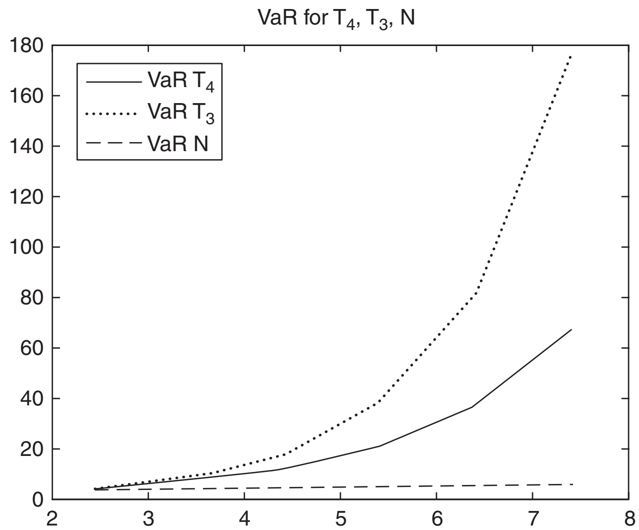

Figure 1.9 Risk Measure VaR for the Student distribution with 3 degrees of freedom, 4 degrees of freedom and the Gaussian distribution

makes us very aware that Non-Gaussian modelling is necessary and we therefore provide the numerical tools to handle this modelling approach.

Figure 1.9 plots the VaR for different probability distributions, namely the Student distributions with 3 and 4 degrees of freedom as well as the normal distribution. We see the very low values for the VaR with respect to the normal distribution. Choosing the Gaussian distribution suggests that the tail risk is very small. Tail events, thus, cannot be neglected. But by choosing another distribution, such as the Student distribution with a small degree of freedom, we observe that the tail risk is significant.

Other risk measures have been suggested, for instance the expected shortfall (CVaR). Figure 1.10 illustrates that for the Gaussian distribution switching from VaR to CVaR has no effect. Both risk measures lead to similar results. However, changing the distribution to a Non-Gaussian distribution shows that switching from VaR to CVaR leads to different risk figures.

The Matlab functions `StuCVaR`, `Stunorm` and `FMinusOne`, illustrated in Figures 1.11 and 1.12, are used to compute the CVaR for the Student distribution. The code displayed in Figure 1.13 is used to compute the VaR.

Another method that may be used here is to identify the key drivers of risk by applying *principal component analysis*, PCA. These key drivers are simulated in stress tests or in some likely scenario. However, when applying PCA care has to be taken since there are several pitfalls. We refer to West, G. (2005) for illustrations and an analysis of the risks involved in PCA.

1.3 IMPLIED VOLATILITY SURFACES AND VOLATILITY DYNAMICS

Time series are not the only source of information for modelling the financial market. There are many liquid instruments for which prices are quoted on information systems or prices

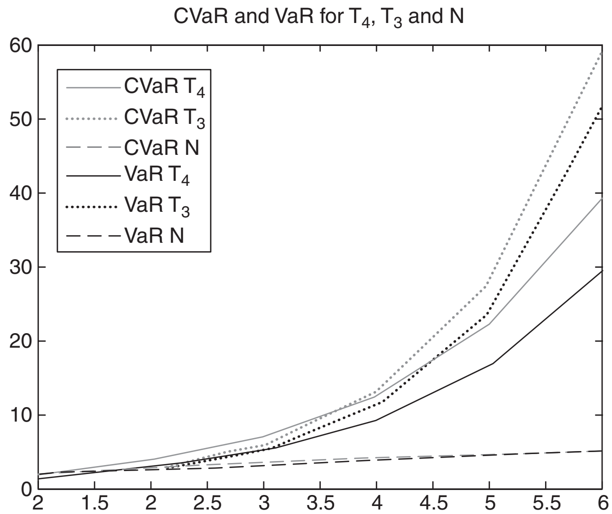

Figure 1.10 Risk Measures CVaR and VaR for the Student distribution with 3 degrees of freedom, 4 degrees of freedom and the Gaussian distribution. Only for the Student distributions do we see that the risk figures are distinct

```
function y = StuCVaR(x,n)
% General method to compute CVaR for Student distribution with n
% degrees of freedom
    f = @(t,n) -(n^(n-2)*(n+t.^2).^(.5-.5*n)*gamma(.5*(n-1))...
        /(2*sqrt(pi)*gamma(.5*n)));
    arg = FMinusOne(x,n);
    y = sqrt((n-2) / n) * f(arg,n)./x;
end
```

Figure 1.11 Matlab code for computing the CVaR for a general Student distribution. $x \in [0, 1]$ and n gives the degrees of freedom

```
function y = FMinusOne(x,n)
    if(x < .5)
        arg = 2 * x;
    else
        arg = 2*(1-x);
    end

    y = sign(x-.5) .* sqrt(n*(1./betaincinv(arg,n/2,.5)-1));
end
```

Figure 1.12 Matlab code for computing the inverse distribution function for a general Student distribution. $x \in [0, 1]$ and n gives the degrees of freedom

```
function y = Stunorm(u,n)
    y = sqrt((n-2)/n)*FMinusOne(u,n);
end
```

Figure 1.13 Matlab code for computing the norm using a general Student distribution. $u \in [0, 1]$ and n gives the degrees of freedom

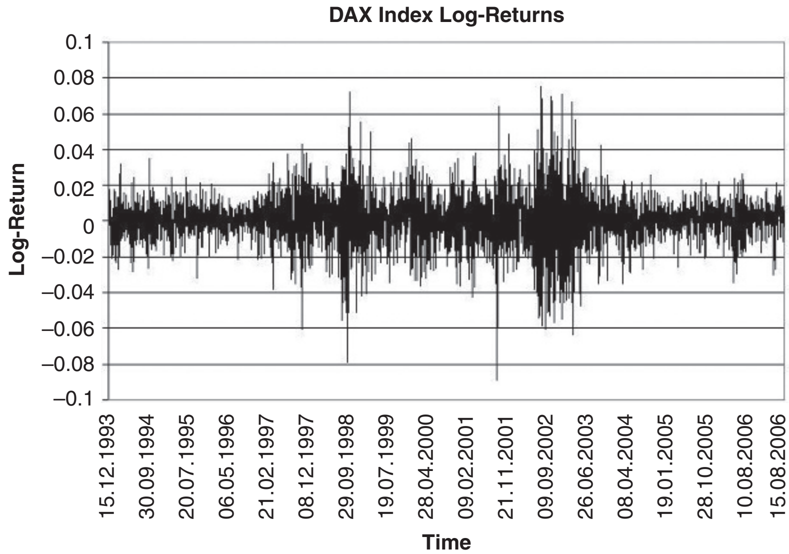

Figure 1.14 Daily logarithmic returns for the DAX index based on time series data from 15.12.1993 until 15.08.2006

can be obtained from brokers. Such quotes are a valuable source of information. Quants try to beg out model parameters from such quotes. This well-known practice is termed the *procedure of calibration*. While we do not discuss the pros and cons of calibration in this book, we do assess the process of calibration for determining some model parameters from given market quotes for a particular model. We wish to find model parameters which most closely match the observed prices. However, using this method completely ignores stability and other modelling issues. For instance, the market quotes can be weighted to assign more weight to liquid options and thus lead to a higher impact when determining the parameters. But the reader may well ask if the calibration makes sense at all, since every day, or even in shorter periods, model parameters change. Despite the criticism of calibration, this is market practice and we do provide the numerical methods and tools to perform calibration. We emphasize that calibrating models needs great care and that all aspects, such as parameter stability or robustness, have to be taken into account.

In this section we consider implied volatility surfaces. Implied volatility is a convenient way to quote option prices.

1.3.1 Is There More than just a Volatility?

Here we show that there is evidence that there are more notions and meanings to the word volatility. The classical meaning of volatility is the one suggested by the Black–Scholes formula.

We can also base the considerations on observed time series data. For example, let us consider Figure 1.14. The figure shows the logarithmic returns of the DAX index based on a time series of daily data from 15.12.1993 to 15.08.2006. We see that there are periods showing very low activity but other regimes where the daily logarithmic returns change very fast. The

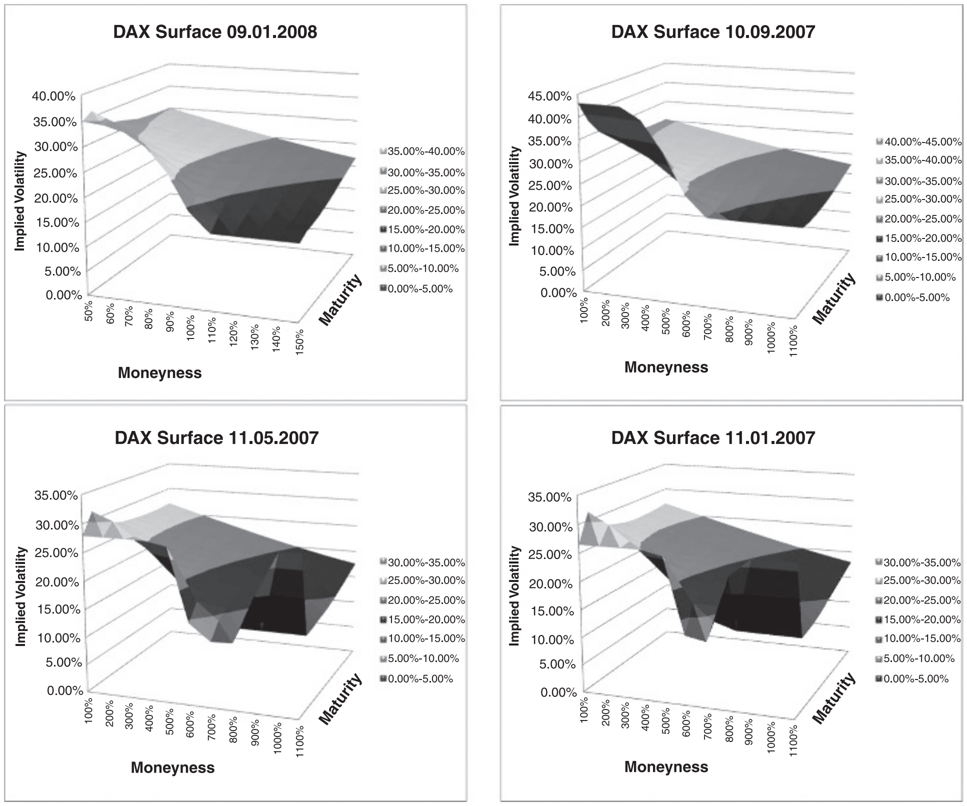

Figure 1.15 DAX Implied Volatility Surfaces for 09.01.2008 (upper left), 10.09.2007 (upper right), 11.05.2007 (lower left) and 11.01.2007 (lower right)

returns also exhibit large absolute values. In fact, it seems that it is impossible to relate a single positive number to the dynamic of the DAX. The dynamic seems to be at least time dependent or even stochastic as in Figure 1.14. Some periods of small moves are followed by small moves, and large moves are followed by large moves. This observation is called *volatility clustering*.

If we consider the implied volatility observed at a given time point, we note that with respect to strike and maturity the implied volatilities do change considerably. Figure 1.15 shows the implied volatility surface for the German DAX index at four different dates, namely 09.01.2008, 10.09.2007, 11.05.2007 and 11.01.2007. We see that not only the height of the surface differs significantly. Comparing the given dates the shapes are also very different. One can see that either a *skew*, high volatility for low strikes and low volatility for high strikes, or a *smile*, high volatility for low and high strikes and low volatilities around the current forward, which is the *ATM strike*, is possible.

This does not fit in terms of aggregating the risk into one single number. In fact, there are many financial products that are very sensitive to changes in the shape of the volatility surface. But, even worse, there are instruments that are sensitive to the movement of the implied volatility surface. We consider such options when we introduce the numerical methods for pricing.

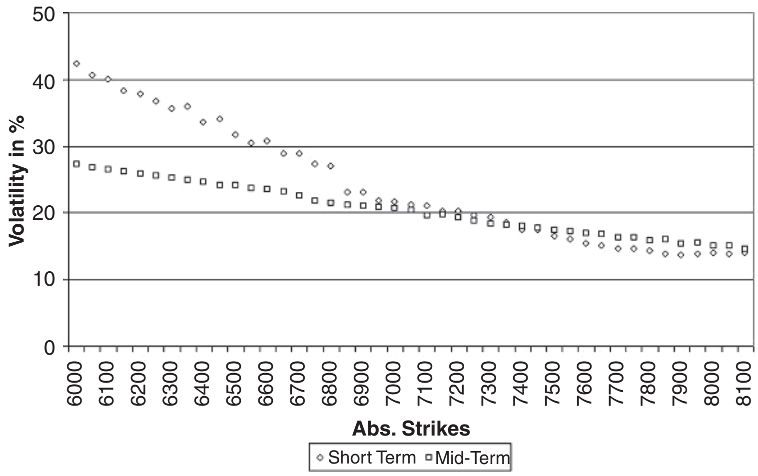

Figure 1.16 A typical shape of the implied volatility structure for DAX options. Plotted are implied volatilities quoted for short-term and mid-term options. The shape suggests that the skew flattens out for large times to maturity

The skew inherent in the implied volatility surface for equity or index options has many causes. We have observed a flat implied volatility surface since the crash in October 1987. A firm's value of equity can be seen as the present value of all its future income plus its assets minus its debt. Assets may have very different relative volatilities than debt, which can give rise to a leverage related skew. Furthermore, supply and demand might be another reason for the skew. Equivalently, downward risk insurance is more desired due to the intrinsic asymmetry of positions in equity. Due to their financial rationale, it is more natural for equity or indices to be held for a long period than for a short one. This makes protection against downward moves more important. Finally, declining stock or index prices are more likely to give rise to massive portfolio re-balancing. Therefore volatility rises if stock or index prices increase. This asymmetry occurs naturally from the existence of thresholds below which positions must be cut unconditionally for regulatory reasons.

The interest rate market smile is caused by the many different market activities of central banks, pension funds or governments. Trading, speculation, hedging or simply investment strategies cause supply and demand for protection against low as well as high interest rate levels. A typical smile shape is the hockey stick.

Foreign exchange rates are quoted in terms of an exchange rate for one currency against another. The smile is caused by anticipating government policies, trade policies and/or hedging activity caused by firms and banks. Foreign investors seek protection against FX rates. Due to the strength of one currency against another the smile can be asymmetric.

Two figures illustrate this phenomenon. First, Figure 1.16 once more illustrates that options having different strikes and different times to maturity are priced using different volatilities.

There is an index that measures realized volatilities. This index is the *VDAX*. Figure 1.17 shows data from the VDAX time series.

Traditionally, smile models have been assessed according to how well they fit market option prices across strikes and maturities. However, the pricing of most recent exotic structures, such as reverse cliquets or Napoleons, is more dependent on the assumptions made for the

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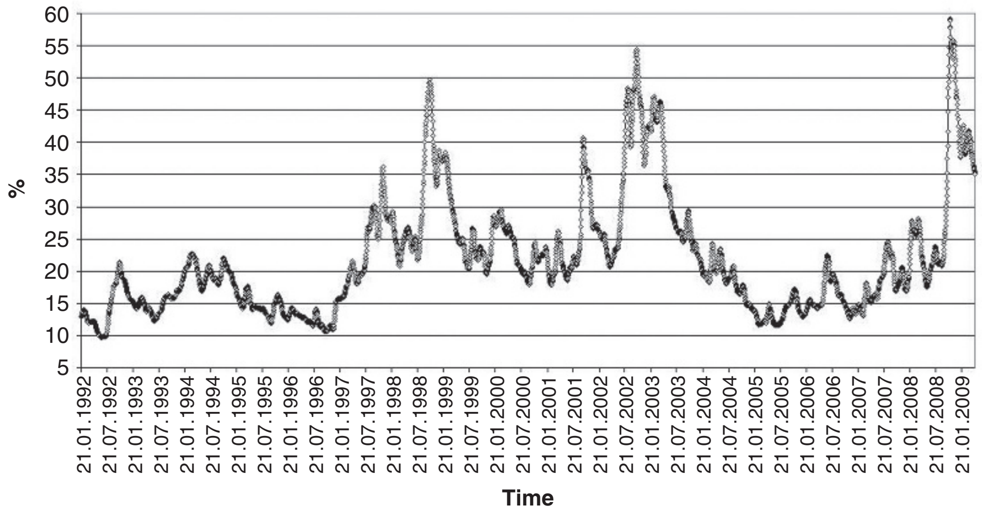

Figure 1.17 Time series for the German Volatility Index VDAX from 21.01.1992 till 21.01.2009. Indicated is the assumption of a time-dependent and even a stochastic volatility

future dynamics of implied volatilities than on today's Vanilla Option prices. We have different assumptions on the smile dynamics, for instance:

- Constant (Black–Scholes).
- Spot smile and spot term structure (Local-Volatility, Merton).
- Moving smile (height, smiling, skewness, ...) (Stochastic Volatility).
- Floating smile property (Lévy).

After gathering information from financial markets we consider different notions for volatility which later lead to financial models based on different stochastic processes. For the different notions of volatility and their connection we refer the reader to Gatheral, J. (2006) and Lee, R. (2002).

1.3.2 Implied Volatility

Let us assume that some financial asset $S(t)$ is modelled by the SDE

$$dS(t) = rS(t)dt + \sigma S(t)dW(t).$$

The parameter σ is the implied volatility. Using σ it is possible to value European Calls and Puts by the Black–Scholes formula. The price of a European Call option, respectively European Put option, is linked by a one-to-one relation to the implied volatility. Thus, option prices can be quoted by just one number regardless of the current spot price.

1.3.3 Time-Dependent Volatility

If we assume that volatility of the underlying price is a deterministic function of time, the process can be defined by the SDE

$$dS(t) = rS(t)dt + \sigma(t)S(t)dW(t),$$

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with $\sigma : \mathbb{R} \to \mathbb{R}^+$ being a positive real-valued function. We define the average volatility, σ_{Av}, by

$$\sigma_{\text{Av}} := \sqrt{\frac{1}{T}\int_0^T \sigma^2(t)dt}.$$

It is well known that $\log(S(t))$ is normal with mean $(r - \frac{\sigma_{\text{Av}}^2}{2})T$ and variance $\sigma_{\text{Av}}T$. A European Call option can be priced using the average volatility. But hence the average volatility is given as an integral expression, the function $\sigma(\cdot)$ cannot be derived from a finite set of European Call option prices. Thus, to restrict the degrees of freedom in a time-dependent volatility model the modeller can use a parametric form of the volatility function $\sigma(\cdot)$.

A further generalization is the time-dependent and spot-dependent volatility.

1.3.4 Stochastic Volatility

Further generalization of the concept of volatility leads to a consideration of *stochastic volatility*. In contrast to the other models volatility by itself is a stochastic process and driven by another Brownian motion.

$$\begin{aligned} dS(t) &= rS(t)dt + f(\sigma(t))dW(t) \\ d\sigma(t) &= \mu(\sigma, t)dt + \alpha(\sigma, t)V(t)dW_2(t) \\ \langle dW_1(t), dW_2(t)\rangle &= \rho dt. \end{aligned}$$

As in the case of time-dependent volatility it is possible to define an average volatility depending on the realization ω:

$$\sigma_{\text{SAv}}(\omega) = \sqrt{\frac{1}{T}\int_0^T \sigma^2(\omega, t)dt}.$$

In contrast to the time-dependent volatility, the expected value of the average volatility is not equal to the implied volatility. If $C(T, K)$ denotes the European Call option price, we have

$$dC(K, T) = e^{-rT}\mathbb{E}\left[-rKH(S(T) - K) + \frac{1}{2}\sigma^2(T)S^2(T)\delta(S(T) - K)\right]dT,$$

with H and δ denoting the *Heavyside* function and the δ distribution.

1.3.5 Volatility from Jumps

There is yet another way for a process to move. Let us take a diffusion process to which we add another stochastic process, which is called the *jump part*. This process does not continuously contribute to the movement but instead there are rare events called *jump times*. In the case of a jump, the jump height is drawn from some probability distribution. The corresponding value is added to the current value. This procedure finally leads to the sample path.

Despite the fact that a process can consist of a diffusion and a jump part, another class of processes has been researched. This class only move by jumps. Jumps are then no longer rare events, but the jumps stochastically vary in size and small jumps contribute to the movement. *Pure jump processes*, as such processes are called, become increasingly important for modelling.

1.3.6 Traders' Rule of Thumb

There are some rules of thumb for modelling volatility. Three of these rules are called the *sticky-strike*, *sticky-moneyness* and the *sticky-delta* rule. Some traders used to adhere strictly to these rules. Therefore, we shall explain them briefly here.

Let us consider a given implied volatility surface V at time t_0. For each time to maturity T, each strike K and level of the underlying S, the function V assigns a value called the *implied volatility*. Mathematically, we can view V as a function

$$\begin{aligned} IV : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ &\to \mathbb{R}^+ \\ (K, T, S) &\mapsto V(K, T, S). \end{aligned}$$

Often we do not consider the current value of the underlying and regard IV as a function of maturity and strike only. Given IV, the sticky-strike rule postulates that for a given time to maturity T the skew varies linearly. Let $v(T)$ denote the slope of the skew that is $v(t) := \frac{\partial C}{\partial K}\big|_{K=S(0)}$. For some function $v_1 : \mathbb{R}^+ \to \mathbb{R}$ the value of the implied volatility is then given

$$IV(K, T) = IV(S(t_0), T) + v_1(t)(S(t_0) - K).$$

First of all, we observe that there is no level dependence of the implied volatility. However, the formula is appealing for equity markets and not too far out of the money strikes. Since we have no level dependence on the spot when modelling the skew we exactly recover the standard Black–Scholes Delta for the options.

Let us now consider the quantity K/S and look at another parametrization of IV. The quantity K/S is called *moneyness*; we will come across this concept and that of log-moneyness, $\log(K/S)$ again later in the book. We wish to introduce some level dependence to V. The rule we now consider is called the *sticky-moneyness rule* and is a little bit more involved. Given IV, the sticky-moneyness rule postulates that if time passes from t_0 to $t_0 + \Delta$ and the asset moves, the volatility of an option for a given strike K sticks to the moneyness. We therefore look at another function $v_2 : \mathbb{R}^+ \to \mathbb{R}$ and consider

$$IV(K, T) = V(S(t_0), T) + v_2(t)(1 - K/S(T))S(t_0).$$

The sticky-delta rule is now an approximation to the sticky–moneyness rule. For strikes not too far away from the current spot level the volatility is approximated by

$$IV(K, T) = V(S(t_0), T) + v_2(t)(S(T) - K).$$

1.3.7 The Risk Neutral Density

The risk neutral density is closely linked to the notion of volatility since European Call or Put options determine (in principle) the *risk neutral density*. For a European Call option we have

$$\begin{aligned} C(K, T) &= e^{-r(T-t)}\mathbb{E}[\max(S(T) - K, 0)|S(t)] \\ &= e^{-r(T-t)} \int_0^\infty \max(S(T) - K, 0) q(S(T)|S(t)) dS(T), \end{aligned}$$

with some probability density q. The probability density that prices all European Call options is called the risk neutral density. With respect to this formula, Breeden, D. and Litzenberger, R. (1978) show that q can be obtained from the continuum of all European Call option prices

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by differentiation

$$\frac{\partial^2 C}{\partial K^2} = e^{-r(T-t)} q(S(T)). \tag{1.3}$$

We use variants of the risk neutral density. Often it is convenient to consider the risk neutral density not with the forward but with the logarithmic moneyness. To this end, let p denote the density and q the density of $\log(S(T)/S(0))$, then, p and q are related by

$$\begin{aligned} p(x) &= q(\log(x/S(0))/x \\ q(x) &= p(S(0)\exp(x))S(0)\exp(x). \end{aligned}$$

In reality there are only quotes available for a discrete set $\mathcal{K} := \{K_1, \ldots, K_N\}$. Thus, further assumptions on strike values $K < K_1$ and $K > K_N$ as well as for strikes K which do not belong to $\mathcal{K}$ but $K_1 < K < K_N$ have to be made. This is directly linked to the issue of using extrapolation and interpolation methods. Some standard interpolation techniques such as linear interpolation are not sufficient for discovering the risk neutral density.

The interpolation method has to be *arbitrage free*. This means that using the interpolated option prices should not lead to arbitrage possibilities as this would lead to unusable volatility structures for the traders and for risk management.

However, having specified extrapolation and interpolation schemes and applying (1.3), one may observe that risk neutral distributions are in general not symmetric and differ significantly from the Gaussian distribution. We discuss several different distributions which we apply for modelling in this book.

A cumulative distribution $F_X(x) := \mathbb{P}(X \leq x)$ is said to obey a fat tail of order α if

$$1 - F_X(x) \sim x^{-\alpha} \text{ as } x \to \infty.$$

This property is linked to the *kurtosis* of the distribution which we consider in the sequel. Let μ be the mean of the distribution. If for the density function

$$f(\mu - x) = f(\mu + x)$$

holds, we say the distribution is symmetric. If this is not the case, the distribution is asymmetric or skewed.

To this end, higher moments become important for analysing time series for financial models. The moments encode properties of the distribution.

The Gaussian distribution, for instance, is completely determined by two parameters. Knowing the mean and the standard deviation or, equivalently, the variance is sufficient to determine the probability distribution. Other distributions with a richer structure can therefore be more realistically applied to financial data showing fat tails, asymmetry and other shapes. For such distributions these features are related to the higher moments. Knowing only the first two moments is not enough information in this case. Below we consider the first four moments and their sampled counterparts, also called *empirical moments*.

The first moment, the *expectation*, is given by

$$\mu = \mathbb{E}[X] \tag{1.4}$$

$$\hat{\mu} = \frac{1}{N}\sum_{j=1}^{N} x_j \tag{1.5}$$

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and the second moment, the *variance* or its square root, the *volatility*

$$\sigma = \sqrt{\mathbb{E}[(X-\mu)^2]} \tag{1.6}$$

$$\hat{\sigma} = \frac{1}{\sqrt{N}}\sqrt{\sum_{j=1}^{N}(x_j-\hat{\mu})^2}. \tag{1.7}$$

Financial time series in general show significant skewness

$$\mathrm{sk} = \frac{\mathbb{E}[(X-\mathbb{E}[X])^3]}{\mathbb{V}[S]^3} \tag{1.8}$$

$$\hat{\mathrm{sk}} = \frac{1}{N}\sum_{j=1}^{N}\frac{(x_j-\hat{\mu})^3}{\hat{\sigma}^3} \tag{1.9}$$

and, finally, kurtosis

$$\mathrm{ku} = \frac{\mathbb{E}[(X-\mathbb{E}[X])^4]}{\mathbb{V}[S]^2} \tag{1.10}$$

$$\hat{\mathrm{ku}} = \frac{1}{N}\sum_{j=1}^{N}\frac{(x_j-\hat{\mu})^4}{\hat{\sigma}^4} \tag{1.11}$$

and therefore are heavily tailed and peaked.

A symmetric distribution, for instance the normal distribution, has skewness 0. If a distribution has fatter tails to the left (right) the skewness is negative (positive).

The normal distribution has kurtosis equal to 3. If the distribution is flatter (higher peaked) than the normal distribution, it has kurtosis smaller (bigger) than 3.

We have reviewed the real world probability density and the risk neutral density. We observed that symmetric, non-fat-tailed distributions often fail to model observed market structures and therefore the risks. To this end we have to apply other distributions and other stochastic processes for modelling apart from the Gaussian model. In the next section we show some financial applications where we wish to apply the more general concepts. To that efficient and robust techniques have to be considered. This is the main topic of the current book.

1.4 APPLICATIONS

Finally, we address the applications we are considering in this book. We wish to use advanced mathematical models for

- Asset Allocation.
- Pricing, Hedging and Risk Management.

Therefore, we discuss these topics in what follows.

1.4.1 Asset Allocation

Suppose a trading desk or a fund manager wishes to create a certain yield. To structure the portfolio several financial assets can be used. Each asset may create a certain yield. But there are also risks, the volatilities, and, furthermore, it may become important how the different assets and asset classes interact with each other. To this end we ask ourselves the question: Is there a way to earn a certain yield and at the same time minimize the risk?

If it is possible to buy and sell securities, options and structured products of different types we have to take into account the distribution of the risk factors. This distribution determines the future value of the portfolio. To obtain the distribution we have to not only estimate parameters describing the movement of a single risk factor but also, moreover, determine the dependencies of the risk factors.

Now, suppose we know the probability distribution but we are faced with another problem: the investment is restricted to certain portfolios. For instance, the fund manager is only allowed to restructure 10% of the portfolio and it must contain 15% equity but not more than 25%. Here we have to solve an optimization with constraints.

In the book we provide measures for the risk and tools to set up the problem as well as applying the optimization.

1.4.2 Pricing, Hedging and Risk Management

Suppose we have a given portfolio consisting of financial products. The portfolio may, for instance, include options, swaps, equities or structured products. This portfolio is exposed to several risk factors. We consider market risk, credit risk and liquidity risk. Then, we have to identify the key risk factors. A *hedge* is an immunization of the portfolio with respect to a given risk factor. For instance, we consider a portfolio consisting only of a European Call option. The standard risk factor which is considered is the spot price of an underlying asset. Thus, a change of the spot price causes a move in the option value.

Let us now consider a hedge against small moves of the underlying. To this end a trader buys or sells a certain amount of the underlying to account for this move. In general we consider two methods for hedging a single option or even a whole trading book:

- Dynamic Hedging.
  Dynamic hedging refers to a trading strategy we eventually apply to adjust a given portfolio to reduce or even cancel out the risk. If we have identified risk factors, we are interested in the sensitivity of the portfolio to a change in this factor. Having some pricing model at hand we can calculate model sensitivities. These sensitivities are also called the *Greeks*. We adjust the portfolio by buying or selling an amount of an instrument corresponding to the risk factor. Thus, when risk factors move, the hedging portfolio has to be changed. Since restructuring the hedge portfolio costs money, we have to decide when and how often we have to re-balance it to reduce the risk.
- Static Hedging.
  Another approach to hedging is the concept of static hedging. A static hedge is a portfolio of financial instruments that approximately or even exactly replicates the payoff of some exotic option or structured product. The static replication does not depend on a certain model but on the structure of the derivatives contracts. But when it comes to pricing the replicating portfolio a model dependence comes in. When we apply static hedging the portfolio is only set up at the beginning and then never altered. Sometimes we decide to hedge only a portion of the risk. The corresponding strategy is a sub-hedge. The opposite of a sub-hedge is a super hedge.

Let us briefly consider static hedging. To illustrate the concept we consider a function

$$\begin{aligned} f : I &\to \mathbb{R} \\ S &\mapsto f(S) \end{aligned}$$

such that f is monotonically increasing and for some number K_0 we have $f(K_0) = 0$. Then we can find an approximate replication in terms of Call option payoffs by

$$f(S) \approx \sum_{i=1}^{N} \omega(K_i)(S - K_i)^+.$$

The weights ω_i, $i = 1, \ldots, N$ can be determined and they are given by

$$\omega_0 = f'(K_0); \omega_1 = f''(K_1 - K_0); \ldots ; \omega_i = f''(K_i)(K_{i+1} - K_i),$$

which in the limit leads to a full replication

$$f(S) = f'(K_0)(S - K_0)^+ + \int_{K_0}^{\infty} f''(K)(S - K)^+ dK.$$

If the function f is monotonically decreasing and $f(K_0) = 0$ we have

$$f(S) \approx \sum_{i=1}^{N} \tilde{\omega}_i(K_i)(K_i - S)^+,$$

with weights

$$\tilde{\omega}_0 = -f'(K_0); \tilde{\omega}_1 = -f''(K_0 - K_1); \ldots ; \tilde{\omega}_N = -f''(K_N)(K_{N-1} - K_N),$$

which than leads to a replication formula applicable for Put payoffs and we find for the limit

$$f(S) = -f'(K_0)(K_0 - S)^+ + \int_{0}^{K_0} f''(K)(K - S)^+ dK.$$

We consider the approach for two different instruments and explain its application. First, we remark that for $f(S) := ((S - K)^+)^2$ we have $2((S - K)^+)^2 \int_K^\infty (S - x)^+ dx$ and $\mathbb{E}[((S - K)^+)^2] = 2 \int_K^\infty \mathbb{E}[(S - x)^+] dx$.

Now, we are in a position to consider some well-known derivatives which can be statically hedged.

- Constant Maturity Swaps.
  We consider the set of time points $T_a, T_{a+1}, \ldots, T_b$ and the swap rate

$$S_{a,b} := \frac{P(t, T_a) - P(t, T_b)}{A_a(t)}; A_a(t) := \sum_{j=a+1}^{b} \tau_j P(t, T_j).$$

  The expectation of the swap rate with respect to the T_i forward measure is

$$\mathbb{E}^{T_i}[S_{a,b}(T_a)] = \frac{A_a(0)}{P(0, T_a + \delta)} \mathbb{E}^A \left[S_{a,b}(T_a)^2 \frac{P(T_A, T_a + \delta)}{A_a(T_a)} \right].$$

  In the final equation the second factor can be approximated and differentiated with respect to the swap rate and, then, replication is applied. This leads to the notion of convexity adjustment for constant maturity swaps.

- Variance Swaps.
  We consider the payoff of a log contract

$$\int_0^T \sigma(t)dt = 2\int_0^T \frac{1}{S(t)}dS(t) - 2\log\left(\frac{S(T)}{S(0)}\right)$$

and using the fact that the quadratic variation is $\left\langle \frac{\int_0^T \sigma^2(t)dt}{T} \right\rangle = \frac{2}{T}\left\langle \log\left(\frac{F}{S(T)}\right)\right\rangle$ we have

$$\frac{2}{T}\left(\int_0^{S(T)} \frac{\langle (K-S(T))^+\rangle}{K^2}dK + \int_F^\infty \frac{\langle (S(T)-K)^+\rangle}{K^2}dK\right)$$

and therefore a method to statically hedge the Variance Swap.

In fact, we cheated when stating the replication formula for CMS. It is market standard to use cash settled swaptions instead of physically delivered swaptions. To this end the replication formula should involve a change of numeraire.

To this end let M and N be two different numeraires with associated measures Q_M and Q_N. The numeraires represent the ones associated with cash settled and physically settled swaptions. Furthermore, let f_r denote the quotient M/N and $Y(t) = f(S(t))$ for some twice differentiable function f. Now, let us consider another twice differentiable function g and we are interested in the values of options with payoffs:

$$g(S(T))(S(T)-k)^+ \text{ and } g(S(T))(k-S(T))^+.$$

Let C_0 and P_0 denote the corresponding values of the options. Then, if we set $h(x) := f/g$ and $\omega := \partial^2/\partial x^2 h f_r$

$$\begin{aligned} V_0(Y(T)) &= f' f_r(K) + C_0(K) - P_0(K) \\ &\quad + \int_0^K \omega(x)P_0(x)dx + \int_K^\infty \omega(x)C_0(x)dx \end{aligned}$$

provided that $K \geq 0$ such that $f(K) = 0$, $g'(K) \neq 0$.

We can now take for f and g the physically settled and the cash settled annuity. The above formula, then, leads to the replication formula for CMS using cash settled swaptions.

Wittke, M. (2011) considers the convexity correction computed via a replication portfolio for CMS. Let us consider the implementation in Matlab. We take, for instance, the SABR model to determine the smile. First, we give the code for computing the weights.

The function `weights` (Figure 1.18) uses the function `annuity`, as illustrated in Figure 1.19.

The weights are multiplied by the payer, respectively receiver swaption prices of the replication portfolio. The corresponding swaptions are valued using a Black formula. The volatility we use for pricing is determined by the SABR model. This model is used to incorporate the observed swaption smile in the market. However, some of the shortcomings of the SABR model and the rule of thumb formula are also considered below.

Now, having the weights and the swaption prices of the replication portfolio, we can use the current forward as the strikes to determine the convexity adjustment. Using the above algorithms, especially the code from Figure 1.20, we have evaluated our first static hedge using Matlab. The above mentioned shortcomings of the SABR model lead to unstable convexity adjustments so we need more advanced methods to robustly and effectively price CMS and CMS options.

```
function [value] = weights(strikes,tenor,yearfrac)

    NStrikes = length(strikes);
    value = zeros(1,NStrikes-1);

    for i = 1:1:NStrikes-1
    wSum = 0;
        for j = 1:1:i
            tmp = (strikes(i+1)-strikes(j))/(strikes(i+1)-strikes(i));
            wSum = wSum + value(1,j)*tmp;
        end
    value(1,i) = (strikes(i+1)-strikes(1)) ...
       /(annuity(strikes(i+1),tenor,yearfrac)* ...
              (strikes(i+1)-strikes(i)))-wSum;
    end

end
```

Figure 1.18 Matlab implementation for the weights used for replicating CMS caplets and floorlets

1.5 GENERAL REMARKS ON NOTATION

Let us consider a stochastic process $S(t)$ as well as the logarithm of S. The process $X(t) := \log(S(t))$ is given by:

$$S(t) = S(0)\exp(\mu(t, S(t))t - \omega(t) + \sigma_L(t, S(t))W(t)) \tag{1.12}$$

$$X(t) = X(0) + (r - d)t - \omega(t) + L(t) \tag{1.13}$$

or equivalently in terms of the corresponding *Stochastic Differential Equations*:

$$dS(t) = \mu(t, S(t))dt + \sigma_L(t, S(t))dW(t) \tag{1.14}$$

$$dX(t) = \tilde{\mu}(t, S(t))dt + \tilde{\sigma}_L(t, S(t))dW(t). \tag{1.15}$$

The process ω is called the *martingale adjustment*. We consider several methods for making an exponential model into a martingale. For the examples we always stick to the subtraction of the mean value to assure the martingale property.

For specific markets either prices or volatilities are quoted. To this end the modeller has to account for the given quotes. For example, we may wish to set up a calibration procedure. If the market quotes implied volatilities but not option prices, it would be reasonable to use an approximation formula or a closed form solution for the quoted volatility instead of a pricing formula. This is reasonable since the price has to be transformed into a volatility, which can be very time consuming. The appendix of this chapter, Section 1.7, gives several market quotes including swaption volatilities, CMS and CMS Spread options. Such quotes can be used for calibration purposes or to verify prices. In general, the modeller has to decide which kinds of market data to choose. There are several types of information available:

```
function value = annuity(x,tenor,yearfrac)
    value = 1./x.*(1-(1./(1+x.*yearfrac).^(tenor./yearfrac)));
end
```

Figure 1.19 Calculating the annuity for determining the weights for CMS caplet and floorlet replication

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```
function [value] = Caplet_Rep_SABR(fs, strike, ...
        maturity, tenor, yearfrac, ...
        alpha, beta, nu, rho)
    w = weights(strike,tenor,yearfrac);         % calc weights
    swaptions = payerSwaptionSABR(fs, strike, ...
        maturity, tenor, yearfrac, ...
        alpha, beta, nu, rho);                  % swaption prices
    value = w*swaptions(1:end-1)';              % caplet
end

function [value] = Floorlet_Rep_SABR(fs, strike, ...
        maturity, tenor, yearfrac, ...
        alpha, beta, nu, rho)
    w = weights(strike,tenor,yearfrac);
% replication weights
    swaptions = receiverSwaptionSABR(fs, strike, ...
        maturity, tenor, yearfrac, ...
        alpha, beta, nu, rho);                      % swaption prices
    value = w*swaptions(1:end-1)';                  % floorlet
end
```

Figure 1.20 Code for implementing the replication method for CMS caplets and floorlets. The functions `payerSwaptionSABR` and `receiverSwaptionSABR` are implemented by using the SABR model

- Historical Data (time series).
- Quoted Option Prices.
- Liquid Options / Illiquid Options.
- Broker Prices for Exotics.

Often there is not a particularly good choice of quotes. Thus, the modeller has to rely on the data that are available. For things which can go wrong by choosing raw market data the reader is referred to Kienitz, J., Wetterau, D. and Wittke, M. (2011).

1.6 SUMMARY AND CONCLUSIONS

In this chapter we have reviewed market observable data from the financial markets. We identified several notions of volatility and we observed that the Gaussian modelling assumption, which is ubiquitous in modelling, is not adequate. Finally, we identified several applications such as asset allocation, pricing and hedging where the application of sophisticated models is needed. We argued that the following steps are crucial for correct modelling:

- Market Data.
- Model Choice.
- Calibration.
- Pricing.
- Hedging / Risk Management.

Assessing the market data and undertaking a statistical analysis on which the model choice is based is crucial. To be able to use complex mathematical models with confidence, appropriate numerical algorithms are necessary. We have written this book to cover such algorithms for a wide range of models. We provide well-known and widely applied models that are capable of modelling the market observable structures. The models can be efficiently and effectively implemented. Our choice here is Matlab, for which we provide source code for all the examples discussed in this book. In this way it is possible to extend the classical applications to much more complex models. We show the impact of parameters for each model and describe all the numerical techniques in detail. Using the source code the reader can try out the models and test if the implementation is efficient and stable enough for proprietary applications.

1.7 APPENDIX – QUOTES

Tables 1.1, 1.2, 1.3 and 1.4 give examples for market quotes for interest rate options, equity options or foreign exchange options. Such market quotes can be applied to infer model parameters, test models and numerical methods.

Table 1.1 Market quotes for the spread over 3M-Euribor for constant maturity swaps (CMS). This spread reflects the convexity adjustment

Swap	2Y Index	5Y Index	10Y Index	20Y Index	30Y Index
5Y	51.7/57.7	100.7/109.7	147.9/157.9	154.6/174.6	139.1/164.1
10Y	45.5/51.5	81.6/90.6	117.0/127.0	107.2/127.2	97.8/122.8
15Y	40.5/46.5	66.2/75.2	91.1/101.1	82.6/102.6	79.2/104.2
20Y	35.6/41.6	56.7/65.7	71.1/91.1	79.8/99.8	77.7/107.7

Table 1.2 Market quotes for CMS spread options quoted as caps (Cap) or floors (Flr)

	FWD	ATM	Flr −0.25	Flr −0.10	Flr 0.00	Cap 0.25	Cap 0.50	Cap 0.75	Cap 1.00	Cap 1.50
1y	1.33	27.4	0.7	0.8	0.9	82.8	64.7	47.1	30.8	8.1
2y	1.28	82.1	5.0	5.8	6.4	188.1	147.4	108.5	73.0	24.1
3y	1.17	150.3	11.3	13.3	14.8	269.7	209.1	152.8	103.0	36.6
4y	1.05	228.6	21.6	25.3	28.3	333.9	256.3	185.8	125.5	46.9
5y	0.96	311.9	35.8	41.9	46.7	389.7	297.3	215.0	145.9	57.3
7y	0.83	490.1	74.9	86.7	95.9	497.3	378.7	275.4	190.6	82.5
10y	0.71	777.4	157.7	178.7	194.9	650.0	498.4	369.0	264.5	131.4
15y	0.50	1293.8	351.7	394.8	427.6	826.4	643.7	490.2	367.2	208.4
20y	0.39	1743.8	533.4	596.6	643.6	995.8	785.4	608.3	465.3	275.8

Table 1.3 Market quotes for implied volatility for the German DAX index. The implied volatilities are quoted for different levels of moneyness K/S and maturity T. We use a model which is calibrated to bid/ask prices of quoted option prices and then calculate an implied volatility surface for pricing

T/M	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
50%	39.29%	36.44%	34.64%	33.52%	32.50%	31.93%	31.57%	31.35%	31.15%	31.05%
60%	37.37%	34.31%	32.68%	31.79%	30.99%	30.59%	30.37%	30.28%	30.17%	30.16%
70%	34.61%	31.90%	30.59%	30.01%	29.46%	29.26%	29.18%	29.21%	29.20%	29.28%
80%	31.13%	29.27%	28.42%	28.20%	27.94%	27.93%	28.00%	28.15%	28.25%	28.42%
90%	27.15%	26.50%	26.22%	26.41%	26.43%	26.62%	26.84%	27.11%	27.31%	27.56%
100%	22.97%	23.70%	24.03%	24.64%	24.97%	25.34%	25.70%	26.10%	26.39%	26.73%
110%	19.06%	20.97%	21.90%	22.94%	23.55%	24.10%	24.60%	25.12%	25.50%	25.91%
120%	17.00%	18.75%	19.98%	21.33%	22.19%	22.91%	23.54%	24.16%	24.63%	25.12%
130%	16.75%	17.50%	18.54%	19.97%	20.97%	21.79%	22.52%	23.25%	23.79%	24.35%
140%	16.75%	17.13%	17.68%	18.93%	19.94%	20.80%	21.60%	22.39%	22.99%	23.60%
150%	16.75%	17.12%	17.31%	18.23%	19.13%	19.97%	20.78%	21.61%	22.25%	22.90%

Table 1.4 Market quotes of the swaption smile

	Receivers					Payers			
	−150	−100	−50	−25	ATM	+25	+50	+100	+150
1m2y	41.42	15.27	5.20	2.23	47.94	−0.71	−0.91	−0.43	0.58
1m5y	32.22	17.48	7.54	3.52	40.64	−2.24	−3.84	−5.44	−5.60
1m10y	21.95	12.80	5.74	2.68	32.34	−1.82	−3.15	−4.45	−4.47
1m20y	22.24	13.68	6.48	3.11	30.90	−2.47	−4.57	−7.45	−8.52
1m30y	31.54	20.48	10.92	5.83	34.12	−3.19	−6.05	−10.4	−12.4
3m2y	39.58	15.43	5.74	2.51	48.12	−1.31	−2.17	−2.98	−3.06
3m5y	20.16	9.94	3.77	1.61	38.64	−0.77	−1.16	−1.15	−0.57
3m10y	16.66	9.35	3.91	1.77	30.21	−1.40	−2.44	−3.57	−3.75
3m20y	17.44	10.38	4.66	2.22	28.32	−1.95	−3.59	−5.83	−6.74
3m30y	20.12	12.17	5.51	2.50	30.90	−2.21	−4.12	−6.92	−8.32
6m2y	32.83	13.90	5.13	2.21	48.13	−1.50	−2.63	−4.05	−4.72
6m5y	21.18	11.10	4.43	1.97	37.33	−1.52	−2.65	−3.92	−4.28
6m10y	14.89	8.32	3.45	1.55	29.18	−1.21	−2.10	−3.04	−3.16
6m20y	14.65	8.60	3.80	1.80	27.23	−1.59	−2.91	−4.72	−5.51
6m30y	15.41	9.17	4.10	1.92	29.75	−1.64	−3.03	−5.07	−6.14
9m2y	28.13	12.23	4.53	1.98	47.18	−1.49	−2.64	−4.18	−5.03
9m5y	16.61	8.75	3.55	1.60	35.62	−1.29	−2.32	−3.71	−4.41
9m10y	13.31	7.45	3.11	1.41	28.58	−1.12	−1.98	−2.98	−3.26
9m20y	12.58	7.38	3.29	1.56	26.61	−1.36	−2.50	−4.17	−5.07
9m30y	13.49	7.94	3.53	1.66	28.67	−1.46	−2.71	−4.62	−5.74
1y2y	24.66	11.11	4.13	1.80	45.97	−1.46	−2.59	−4.12	−4.96
1y5y	14.98	7.97	3.25	1.47	34.41	−1.19	−2.14	−3.43	−4.11
1y10y	12.13	6.85	2.86	1.28	27.89	−1.05	−1.88	−2.94	−3.39
1y20y	11.92	6.96	3.03	1.40	26.05	−1.20	−2.19	−3.47	−4.07
1y30y	12.89	7.54	3.32	1.55	27.90	−1.35	−2.50	−4.22	−5.19
2y2y	14.15	7.16	2.80	1.24	37.32	−1.11	−1.99	−3.21	−3.84
2y5y	11.08	6.11	2.52	1.14	29.49	−0.94	−1.68	−2.71	−3.24
2y10y	9.23	5.27	2.24	1.02	25.35	−0.85	−1.54	−2.58	−3.30
2y20y	9.25	5.37	2.31	1.06	24.32	−0.90	−1.66	−2.75	−3.29
2y30y	10.24	5.94	2.63	1.23	25.74	−1.05	−1.92	−3.25	−4.00
5y2y	7.69	4.28	1.80	0.80	24.38	−0.61	−1.06	−1.52	−1.58
5y5y	7.33	4.14	1.74	0.79	21.97	−0.62	−1.09	−1.71	−1.91
5y10y	6.83	3.93	1.69	0.78	20.71	−0.64	−1.14	−1.77	−2.02
5y20y	8.04	4.57	1.91	0.86	20.90	−0.66	−1.16	−1.71	−1.79
5y30y	8.27	4.70	1.99	0.91	22.19	−0.78	−1.39	−2.13	−2.38
10y2y	4.99	2.89	1.28	0.58	18.15	−0.46	−0.82	−1.24	−1.36
10y5y	5.28	3.02	1.28	0.58	17.92	−0.47	−0.83	−1.28	−1.44
10y10y	6.54	3.71	1.52	0.65	18.81	−0.49	−0.87	−1.33	−1.46
10y20y	7.30	4.09	1.68	0.75	19.44	−0.58	−1.01	−1.50	−1.62
10y30y	7.70	4.31	1.78	0.80	19.65	−0.63	−1.10	−1.63	−1.75